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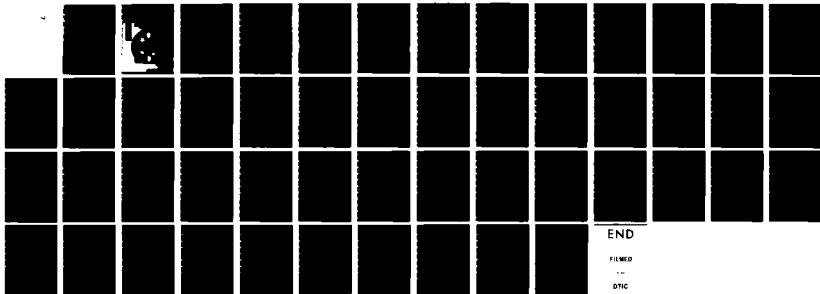
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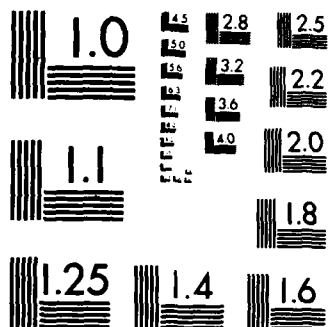
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and

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Division of Applied Mathematics  
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Providence, Rhode Island 02912

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# ABSTRACT

For a transverse homoclinic orbit  $\gamma$  of a mapping (not necessarily invertible) on a Banach space, it is shown that the mapping restricted to orbits near  $\gamma$  is equivalent to the shift automorphism on doubly infinite sequences on finitely many symbols. Implications of this result for the Poincaré map of semiflows are given.



# §1. Introduction.

If  $O$  is an hyperbolic fixed point of a diffeomorphism  $F \in C^k(\mathbb{R}^n)$ ,  $k \geq 1$ ,  $n \geq 2$ , and  $W^S, W^U$  are the stable and unstable manifolds of  $O$ , then  $q \in W^S \cap W^U$ ,  $q \neq O$ , is said to be transverse homoclinic to  $O$  if  $W^U$  is transversal to  $W^S$  at  $q$ ,  $W^U \not\parallel W^S$ . The orbit  $\gamma(q) = \{F^{nq}, n \in \mathbb{N}; \text{ set of integers}\}$  through  $q$  is called a transverse homoclinic orbit asymptotic to  $O$ .

Poincaré was well aware of the fact that the existence of transverse homoclinic orbits implied that the flow defined by  $F$  would be very complicated in a neighborhood of  $q$ . Birkhoff proved that there must be infinitely many periodic points near  $q$ . Smale [15, 16] showed that there was an integer  $k$  and an invariant set  $I$  near  $q$  of  $F^k$  such that  $F^k$  restricted to  $I$  was equivalent to the shift map  $\sigma$  on the set of doubly infinite sequences on two symbols (see, also, Moser [11], Palmer [13]). Silnikov [14] discussed the set of all orbits of  $F$  that remain in a small neighborhood of  $\gamma(q)$ . He then showed that  $F$  on certain subsets of these solutions was equivalent to the shift map  $\sigma$  on the set of doubly infinite sequences on infinitely many symbols.

Our objective in this paper is to generalize these results to the case of  $F \in C^k(X)$ , where  $X$  is a Banach space and  $F$  is not necessarily a diffeomorphism. For a hyperbolic fixed point  $O$  of  $F$ , the local stable set  $W_{loc}^S$  and local unstable set  $W_{loc}^U$  of  $O$  are  $C^k$  manifolds (a proof is given below for completeness). However, the behavior of the global stable set  $W^S$  and unstable set  $W^U$  may not have a nice manifold structure. Even in the case where  $W^U$  is finite dimensional, the local dimension may vary with the point on  $W^U$ . This necessitates hypotheses on  $W^U$  even to define

a transverse homoclinic orbit. Under an appropriate hypothesis on  $W^u$  (there is an immersion from  $W_{loc}^u \times N$  into  $W^u$  which covers  $\gamma(q)$ ), a transverse homoclinic orbit is defined and it is shown that the results of Sil'nikov [14] and Smale [15, 16] are valid. The main theorem is stated and proved in Section 5. The proof is a revised version of the horseshoe argument (see [2], [12]). Holmes and Marsden [6] have also used the properties of horseshoes in the equations of a forced beam. Chaotic motion is discussed in Section 7. The implications for the Poincare map for flows are given in Section 8. Applications to retarded functional differential equations will appear elsewhere.

## §2. Notations and preliminaries.

Let  $X, Y$  and  $Z$  denote Banach spaces. If  $U$  is an open set in  $X$ , then  $C^k(U, Y)$  is the usual space of functions mapping  $\bar{U}$  into  $Y$  which are continuous and bounded together with derivatives up through order  $k$ . The norm in this space is the supremum of all these derivatives. We also let  $C^k(X) = C^k(X, X)$ . The symbol  $(N^-)(N^+)N$  will denote the (non-positive integers) (nonnegative integers) integers. By a submanifold of a Banach space  $Z$ , we mean a regular submanifold (locally expressed as the graph of a  $C^1$  map from  $X$  into  $Y$  where  $Z = X \oplus Y$  is a splitting of Banach spaces).

If  $S$  is a topological space, we let  $\prod_N S$  be the infinite product space with the product topology. An element  $\tau \in \prod_N S$  is a map  $\tau: N \rightarrow S$ . Define  $\sigma: \prod_N S \rightarrow \prod_N S$  as the shift map,  $\tau_1 = \sigma\tau$ ,  $\tau_1(n) = \tau(n+1)$ ,  $n \in N$ . If  $F \in C^0(S, S)$ , a trajectory of  $F$  is a map  $\tau \in \prod_N S$  such that  $\prod F(\tau) = \sigma(\tau)$ , where  $\prod F: \prod_N S \rightarrow \prod_N S$  is defined as  $\tau_1 = \prod F(\tau)$ ,  $\tau_1(n) = F(\tau(n))$ ,  $n \in N$ . Obviously  $\prod F$  is continuous and the set of all the trajectories of  $F$  form

a closed subset  $\Pi_N S$ , which is a topological subspace with the topology induced from  $\Pi_N S$ . In a similar way, one defines respectively a positive (negative) trajectory by a map  $\tau^+(\tau^-)$ . A (positive orbit) (negative orbit) (orbit) will be the range of  $(\tau^+)(\tau^-)(\tau)$  and will be denoted by  $(O_{\tau^+})(O_{\tau^-})(O_{\tau})$ . For  $\tau \in \Pi_N S$ , let  $s_n = \tau(n)$ , and write  $\tau = (\dots, s_{-2}, s_{-1})[s_0, s_1, \dots)$  to indicate that  $\tau(0) = s_0$ . Thus  $\tau_1 = \sigma\tau$  is denoted by  $\tau_1 = (\dots, s_{-2}, s_{-1}, s_0)[s_1, s_2, \dots)$ . And, in this notation,  $\Pi F(\tau) = (\dots, Fs_{-2}, Fs_{-1})[Fs_0, Fs_1, \dots)$ . We shall use  $\tau[i, j]$ ,  $i < j$  integers, to denote the restriction of  $\tau$  to an interval  $[i, j]$ .

Let  $\sim$  be an equivalence relation defined in the topological space  $S$ . For any  $s \in S$ ,  $[s] = \{s_1 : s_1 \sim s\}$  is said to be the equivalence class of  $s$ . The quotient space  $S/\sim = \{[s] : s \in S\}$  is defined with the quotient topology. For a subset  $Q \subset S$ , define  $[Q] = \{[s] : s \in Q\}$  as the equivalence class of  $Q$ .

Suppose  $O$  is a fixed point of  $F \in C^k(X)$ ,  $k \geq 1$ . The fixed point  $O$  is hyperbolic if  $\sigma(DF(O)) \cap \{|\lambda| = 1\} = \emptyset$ , where  $\sigma(A)$  denotes the spectrum of a linear operator  $A$ . The unstable set  $W^u(O)$  and the stable set  $W^s(O)$  of a fixed point  $O$  of  $F$  are defined by

$$W^u(O) = \bigcup \{\text{negative orbits } O_{\tau^-} \text{ of } F : \tau^-(n) \rightarrow O \text{ as } n \rightarrow -\infty\},$$

$$W^s(O) = \bigcup \{\text{positive orbits } O_{\tau^+} \text{ of } F : \tau^+(n) \rightarrow O \text{ as } n \rightarrow +\infty\}.$$

The local unstable and stable sets are defined respectively by

$$W^u(O, U) = \bigcup \{\text{negative orbits } O_{\tau^-} \text{ of } F : O_{\tau^-} \subset W^u(O) \cap U\},$$

$$W^s(O, U) = \bigcup \{\text{positive orbits } O_{\tau^+} \text{ of } F : O_{\tau^+} \subset W^s(O) \cap U\},$$

where  $U$  is an open set containing  $0$ . We use the notation  $W_{loc}^u(0)$ ,  $W_{loc}^s(0)$ , for  $W^u(0,U)$ ,  $W^s(0,U)$  if  $U$  is not relevant to the problem

If  $F$  is a diffeomorphism, one can always consider complete orbits in the definition of  $W^s(0)$ . Furthermore,  $W^u(0)$ ,  $W^s(0)$  are  $C^k$  immersed submanifolds of  $X$  [5]. In particular, if the dimension is finite, then the dimension must be the same at every point. The following examples illustrate the differences that can occur with maps.

Example 2.1.  $F \in C^k(\mathbb{R}^2)$ ,  $k \geq 1$ ,  $F(x,y) = (0,2y)$ . For this case, the only fixed point is the origin  $0$  and  $W^s(0) = \{y = 0\}$ ,  $W^u(0) = \{x = 0\}$ . The map  $F^{-1}$  is only defined on  $W^u(0)$  and is single valued only if the range is restricted to  $W^u(0)$ .

Example 2.2. We construct a delay differential equation with a hyperbolic equilibrium point having a two-dimensional local unstable manifold. The unstable manifold collapses into a smooth one-dimensional manifold along one of the trajectories, a phenomenon that could not happen in ordinary differential equations. The time one map for this example will have the property that the dimension of the unstable manifold is not the same at every point.

Consider the delay equation

$$(2.1) \quad x(t) = \alpha(x(t))x(t) + \beta(x(t))x(t-1),$$

where  $x \in \mathbb{R}$ ,  $\alpha(x)$  and  $\beta(x)$  are defined as

$$(\alpha(x), \beta(x)) = \begin{cases} (\frac{2c-1}{c-1}, -\frac{c^2}{c-1}) & , \quad |x| \leq 1; \\ (1, 0) & , \quad |x| \geq 2; \\ c^\infty \text{ functions of } x, \text{ satisfying that} \\ 1 = \alpha(x) + \beta(x)c^{-1} \text{ when } 1 \leq |x| \leq 2. \\ \text{Also, } \alpha(x) \text{ and } \beta(x) \in C^\infty(\mathbb{R}). \end{cases}$$

The origin 0 is an equilibrium point of (2.1). Equation (2.1) is linear in a neighborhood of 0 and has  $\lambda_1 = 1$  and  $\lambda_2 = 2$  as the positive characteristic values. All the other characteristic values have negative real parts. Thus (see [3]), there is a neighborhood  $U$  of 0 such that  $\dim. W^u(0, U) = 2$ . Let  $x(t) = \varepsilon e^t$  be a solution issuing from  $W^u(0, U)$ . For some large  $\tilde{t} > 0$  we have  $\inf_{-1 \leq \theta \leq 0} |X_{\tilde{t}}(\theta)| > 2$ , and in a neighborhood of  $X_{\tilde{t}}$ , (2.1) becomes  $\dot{x}(t) = x(t)$ . Let  $\phi \in C[-1, 0]$  be in a small neighborhood of  $X_{\tilde{t}}$  and suppose that there is a solution passing through  $\phi$  in the negative direction. It is easy to see that  $\phi(\eta) = \eta e^{\tilde{t} + \eta}$  with  $\eta$  near  $\varepsilon$ . Therefore, the unstable set in this neighborhood of  $X_{\tilde{t}}$  is a smooth manifold but of dimension 1.

Take the time one map  $F = T(1)$  of the solution map  $T(t)$  of (2.1). We have an example with the property that the hyperbolic fixed point 0 of  $F$  has a local two dimensional unstable manifold which collapses into a one dimensional manifold.

Suppose  $F \in C^k(X)$ ,  $k \geq 1$  and 0 is an hyperbolic fixed point of  $F$ . We shall prove that  $W_{loc}^u(0)$  and  $W_{loc}^s(0)$  are submanifolds in §3. An orbit  $O_\tau$  is an homoclinic orbit asymptotic to a fixed point 0 of  $F$  if  $O_\tau \subset W^u(0) \cap W^s(0)$  and  $O_\tau \neq \{0\}$ . An homoclinic orbit  $O_\tau$  asymptotic to a fixed point 0 of  $F$  is said to be a transverse homoclinic orbit if

- 1) 0 is an hyperbolic fixed point;
- 2) for any sufficiently large pair of integers  $i, j > 0$ , such that  $\tau(-i) \in W_{loc}^u(0)$  and  $\tau(j) \in W_{loc}^s(0)$ ,  $F^{i+j}$  sends a disc in  $W_{loc}^u(0)$  containing  $\tau(-i)$  diffeomorphically onto its image which is transverse to  $W_{loc}^s(0)$  at  $\tau(j)$ .

Notice that  $W^u(0), W^s(0)$  may not have a manifold structure even in a small neighborhood of  $O_\tau$ . However, condition 2) implies that we can attach to each  $\tau(k) \in O_\tau, k \in \mathbb{N}$ , small pieces of submanifolds  $W_{loc}^u(\tau(k)) \subset W^u(0)$  and  $W_{loc}^s(\tau(k)) \subset W^s(0)$  diffeomorphic to  $W_{loc}^u(0)$  and  $W_{loc}^s(0)$ , respectively, and such that

$$(2.2) \quad W_{loc}^u(\tau(k)) \cap W_{loc}^s(\tau(k)) = \emptyset \quad \text{at } \tau(k) \in O_\tau.$$

Furthermore,  $FW_{loc}^u(\tau(k-1)) \supset W_{loc}^u(\tau(k))$  and  $FW_{loc}^s(\tau(k)) \subset W_{loc}^s(\tau(k+1))$ . This can be done as follows. If  $i, j$  are given as in condition 2), then  $W_{loc}^u(\tau(k)) = W_{loc}^u(0), k \leq -i$  and  $W_{loc}^s(\tau(k)) = W_{loc}^s(0), k \geq j$ .  $W_{loc}^u(\tau(k)), k \geq j$  is defined as a disc in  $F^{k+i}W_{loc}^u(\tau(-i))$ , diffeomorphic to  $W_{loc}^u(0)$  by 2). For  $-i < k < j$ ,  $F^{k+i}W_{loc}^u(\tau(-i))$  still contains a disc covering  $\tau(k)$ , and shall be defined as  $W_{loc}^u(\tau(k))$ , since  $(F^{i+j})^{-1}F^{j-k}$  is the inverse of  $F^{k+i}$  by 2).  $W_{loc}^s(\tau(k)), j > k$ , can be obtained by considering the transversality of  $F^{j-k}$  to  $W_{loc}^s(\tau(j))$  and (2.2) follows similarly. Therefore, there is an immersion from  $W_{loc}^u(0) \times \mathbb{N}$  into  $W^u(0)$  and an immersion from  $W_{loc}^s(0) \times \mathbb{N}$  into  $W^s(0)$ . Both cover  $O_\tau$  but are not necessarily injective. Briefly, we say that  $W^u(0)$  is transverse to  $W^s(0)$  along  $O_\tau$  if no ambiguity can arise.

Example 2.3. Let us consider the interval map  $F : [0,1] \rightarrow [0,1], F(x) = \mu x(1-x)$   $0 < \mu \leq 4$ . The map  $F$  is not invertible and has a fixed point  $x_0 = 1 - \frac{1}{\mu}, \mu > 1$ , which is hyperbolic if  $\mu \neq 3$ . When  $\mu = 4$ , an homoclinic orbit is plotted in Figure 2.1, which hits  $x_0$  after a finite number of iterates of  $F$ , an observation previously made by Block [1]. It is easy to check that the homoclinic orbit is transverse.

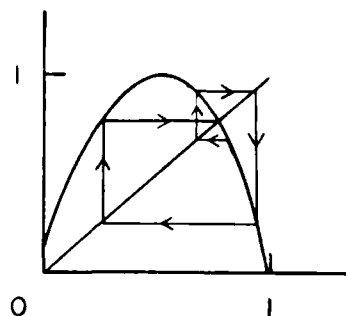


Figure 2.1

Example 2.3 is a special case of snap-back repellers defined by Marotto [10] which will be discussed later.

### §3. Stable and unstable manifolds.

In this section, we state and prove the existence of local stable and unstable manifolds  $W_{loc}^s(0)$  and  $W_{loc}^u(0)$  of a hyperbolic fixed point of a map. The existence of the local stable manifold follows from [7] with very little change needed. For a diffeomorphism  $F$ , the existence of the local unstable manifold follows from the existence of the local stable manifold of  $F^{-1}$ . However, if  $F$  is noninvertible, a direct proof for the existence of the local unstable manifold is needed (see [5]). In spite of the fact that the result may be known to some people, we give the proof for completeness.

#### Theorem 3.1.

Let  $X, Y$  and  $Z = X \times Y$  be Banach spaces and  $A, B$  be linear continuous maps in  $X$  and  $Y$  respectively, with  $\sigma(A) < 1$  and  $\sigma(B) > 1$ . Suppose that  $\|A\|, \|B^{-1}\| \leq \lambda$  for some constant  $0 < \lambda < 1$ . Suppose  $U$  is an open

neighborhood of 0 in Z and  $f_1 : U \rightarrow X$ ,  $f_2 : U \rightarrow Y$  are  $C^k$  ( $k \geq 1$ ) maps with  $f_i(0) = 0$ ,  $Df_i(0) = 0$ ,  $i = 1, 2$ . Consider  $F : U \rightarrow Z$ ,

$$(3.1) \quad F : \begin{cases} x_1 = Ax_0 + f_1(x_0, y_0), \\ (3.2) \quad y_1 = By_0 + f_2(x_0, y_0). \end{cases}$$

Then there exist open balls  $C_1, D_1$  centered at 0 in  $X, Y$  respectively, and a unique  $C^k$  map  $h_1 : C_1 \rightarrow D_1$  with  $h_1(0) = 0$ ,  $Dh_1(0) = 0$  such that

$$F(\text{graph } h_1) \subset \text{graph } h_1.$$

The restriction of  $F$  to graph  $h_1$  is a contraction. Moreover, if  $F^n(z) \in C_1 \times D_1$  for  $n \geq 0$ ,  $z \in \text{graph } h_1$ .

There also exist open balls  $C_2, D_2$  centered at 0 in  $X, Y$  respectively, and a unique  $C^k$  map  $h_2 : D_2 \rightarrow C_2$  with  $h_2(0) = 0$ ,  $Dh_2(0) = 0$  such that the restriction of  $F^{-1}$  from graph  $h_2$  into itself is a well-defined single valued  $C^k$  contraction; thus, a diffeomorphism onto  $F^{-1}(\text{graph } h_2)$  with the inverse  $F$  as an expansion. Moreover, if  $z \in C_2 \times D_2$  and the negatively infinite trajectory  $F^{-n}(z) \in C_2 \times D_2$ ,  $n \geq 0$  exists,  $z \in \text{graph } h_2$ .

For the proof of the last part of the theorem, we consider the Banach space  $\ell$  of the bounded, negatively infinite sequences in  $Z$ ; that is,  $\ell = \{z_{-i}, i \geq 0\}$ , with the norm  $\| \{z_{-i}\} \|_{\ell} = \sup_{i \geq 0} |z_{-i}|_Z$ . Suppose  $g \in C^r(z)$  with all the derivatives being bounded in any bounded set of  $Z$ . The map  $\Pi g : \ell \rightarrow \ell$  is defined as  $\Pi g(z)(-i) = g(z(-i))$ ,  $i \geq 0$  for  $\{z_{-i}\} \in \ell$ . Unfortunately, since continuity does not imply uniform continuity in infinite dimensional Banach spaces,  $\Pi g$  is not  $C^r$  even for  $r = 0$ . The remedy is to consider a subspace  $\ell_0 \subset \ell$ ,  $\{z_{-i}\} \in \ell_0$  if and only if  $z_{-i} \rightarrow 0$  as  $i \rightarrow \infty$ .



The following lemma is very elementary and can be easily proved by induction, but works as well as the lemmas in [7], [8] for composition maps.

Lemma 3.2.

Let  $g : Z \rightarrow Z$ ,  $g \in C^r$  and  $g(0) = 0$ . Then  $\Pi g : \lambda_0 \rightarrow \lambda_0$  is  $C^r$  and  $(\Pi g)^{(k)} = \Pi g^{(k)}$ ,  $k \leq r$ .

Proof of Theorem 3.1.

For any  $\epsilon > 0$  and any Banach space  $E$ , let  $B_\epsilon^E = \{x \in E : |x| < \epsilon\}$ . For  $\epsilon > 0$  sufficiently small and any  $y \in B_\epsilon^Y$ ,  $\gamma \in B_\epsilon^{\lambda_0}$ , define

$$(3.3) \quad G(y, \gamma)(-n) = \gamma(-n) - \left( \sum_{i=n+1}^{\infty} A^{i-n-1} f_1(\gamma(-i)) \right), \quad B^{-n} y - \sum_{i=1}^n B^{-n-1+i} f_2(\gamma(-i))$$

It is not difficult to show that  $G(y, \gamma)(-n) \rightarrow 0$  as  $n \rightarrow \infty$ . Thus,  $G : B_\epsilon^Y \times B_\epsilon^{\lambda_0} \rightarrow \lambda_0$ . Lemma 3.2 implies that  $G \in C^r$ . It is clear that  $G(0, 0) = 0$ . Applying the Implicit Function Theorem to the equation

$$(3.4) \quad G(y, \gamma) = 0$$

in a neighborhood of  $y = 0$ ,  $\gamma = 0$ , we have a unique  $C^r$  map  $\phi : B_{\epsilon_1}^Y \rightarrow B_{\epsilon_2}^{\lambda_0}$ ,  $\phi(0) = 0$ , for some  $\epsilon_1, \epsilon_2 > 0$  which solves (3.4) as  $\gamma = \phi(y)$  in  $B_{\epsilon_1}^Y \times B_{\epsilon_2}^{\lambda_0}$ . Let  $P : \lambda_0 \rightarrow Z$  be the projection taking  $\gamma$  to  $\gamma(0)$ ,  $h_2 : B_{\epsilon_1}^Y \rightarrow X$  defined as

$$\begin{aligned} P\phi(y) &= \phi(y)(0) = \left( \sum_{i=1}^{\infty} A^{i-1} f_1(\phi(y)(-i)) \right), y \\ &= (h_2(y), y) \end{aligned}$$

is  $C^r$  with  $h_2(0) = 0$ . The Implicit Function Theorem also enables us to compute  $D\phi(0)$  by computing  $DG(0, 0)$  and thus, conclude that  $Dh_2(0) = 0$ . It is easy to check, from (3.3), that

$$\phi(y)(-n) = F(\phi(y)(-n-1)), \quad n \geq 0.$$

We have obtained that for  $y_0 \in B_{\varepsilon_1}^Y$ ,  $x_0 = h_2(y_0)$ ,  $z_0 = (x_0, y_0)$ , there exists  $z_{-i} \in F^{-i}z_0$ ,  $i \geq 0$  defined as  $z_{-i} = \phi(y_0)(-i)$  and  $|z_{-i}| < \varepsilon_2$ . Since  $\phi$  is continuous, there exists  $\varepsilon_3 < \varepsilon_1$  such that  $y_0 \in B_{\varepsilon_3}^Y$  implies  $|z_{-i}| < \varepsilon_1$ , especially  $|y_{-1}| \leq \varepsilon_1$ . We shall see very soon (see (i), (ii) below) that

$$G(y_{-1}, \{z_{-i-1}, i \geq 0\}) = 0$$

Thus,  $y_{-1} \in B_{\varepsilon_1}^Y$  and  $x_{-1} = h_2(y_{-1})$ . From

$$y_0 = By_{-1} + f_2(h_2(y_{-1}), y_{-1}),$$

using the Implicit Function Theorem, one concludes that, if  $\varepsilon_1$  is sufficiently small,  $|y_{-1}| \leq \tilde{\lambda}|y_0|$ ,  $0 < \tilde{\lambda} < 1$ , and, thus,  $y_{-1} \in B_{\varepsilon_3}^Y$ . This completes the proof that  $F^{-1}$  is a contraction on graph  $h_2$ ,  $|y| < \varepsilon_3$ .

Let  $C_2, D_2$  be open balls in  $X, Y$  such that  $C_2 \times D_2 \subset B_{\varepsilon_2}^Z$ ,  $D_2 \subset B_{\varepsilon_3}^Y$  and  $h_2(D_2) \subset C_2$ . Then the restriction of  $h_2$  on  $D_2$  satisfies all the assertions except that we have to verify that

if  $\{z_{-i}, i \geq 0\}$  is a negatively infinite trajectory in  $B_{\varepsilon_2}^Z$ , then

$$(i) \quad \{z_{-i}, i \geq 0\} \in \ell_0;$$

$$(ii) \quad G(y_0, \{z_{-i}\}) = 0.$$

For any  $\theta > 0$ , there exists  $\varepsilon_2 > 0$  such that  $\|Df_1\|, \|Df_2\| \leq \theta$  if  $|z| \leq \varepsilon_2$ . Let  $\{z_{-i}, i \geq 0\}$  be a negatively infinite trajectory in  $B_{\varepsilon_2}^Z$ . By induction

$$\begin{cases} x_{-i} = \Lambda^k x_{-i-k} + \Lambda^{k-1} f_1(z_{0i-k}) + \dots + f_1(z_{-i-1}), \\ y_{-i} = B^{-i} y_0 - B^{-i} f_2(z_{-1}) - \dots - B^{-1} f_2(z_{-i}). \end{cases}$$

Let  $k \rightarrow \infty$ ,

$$\begin{aligned} x_{-i} &= \sum_{j=0}^{\infty} A^j f_1(z_{-i-j-1}), \\ y_{-i} &= B^{-i} y_0 - \sum_{j=1}^i B^{-i+j-1} f_2(z_{-j}), \end{aligned}$$

Then,

$$(3.5) \quad |z_{-i}| \leq \lambda^i |y_0| + \theta \sum_{j=1}^i \lambda^{i-j+1} |z_{-j}| + \theta \sum_{j=0}^{\infty} \lambda^i |z_{-i-j-1}|.$$

Suppose  $\delta = \overline{\lim}_{i \rightarrow \infty} |z_{-i}| > 0$ , then for any  $\xi > 1$ , there exists  $i_0 > 0$  such that  $|z_{-i}| \leq \xi \delta$  for  $i \geq i_0$ , and

$$(3.6) \quad |z_{-i}| \leq \lambda^i |y_0| + \theta \lambda^i \sum_{j=1}^{i_0} \lambda^{-j+1} |z_{-j}| + \frac{2\theta}{1-\lambda} \cdot \xi \cdot \delta.$$

If  $\frac{2\theta}{1-\lambda} < 1$ , we can choose  $\xi > 1$  such that  $\frac{2\theta}{1-\lambda} \cdot \xi < 1$ . Let  $i \rightarrow \infty$  in (3.6),  $\overline{\lim}_{i \rightarrow \infty} |z_{-i}| \leq \frac{2\theta}{1-\lambda} \cdot \xi \cdot \delta$ . The contradiction shows that  $\delta = 0$ . Therefore,  $\{z_{-i}, i \geq 0\} \in \ell_0$ , together with (3.5) imply (ii).

§4. Some basic lemmas. Consider  $F : Z \rightarrow Z$  defined as (3.1) and (3.2).

Assume all the hypotheses of Theorem 3.1. By a  $C^1$  change of variable, we assume that the local stable and unstable manifolds are flat, i.e.,  $W_{loc}^s(0) = \{y = 0\}$  and  $W_{loc}^u(0) = \{x = 0\}$ . Thus, in addition to the hypotheses in Theorem 3.1, we assume that  $f_1(0, y) = 0$  and  $f_2(x, 0) = 0$ . Consequently,

$$(4.1) \quad f_{1y}(0, y) = 0$$

$$(4.2) \quad f_{2x}(x, 0) = 0$$

A closed  $\varepsilon$ -ball in a Banach space  $E$  with center zero is denoted by  $\overline{B}_\varepsilon^E$ .

For any  $\theta > 0$ , we choose  $\varepsilon > 0$  so small such that  $|Df_1|, |Df_2| \leq \theta$  in

$\overline{B_\varepsilon^z}$ . We assume that  $W_{loc}^u(0), W_{loc}^s(0)$  is contained in  $\overline{B_\varepsilon^z}$  and

$$(4.3) \quad \lambda + \vartheta < 1.$$

Definition 4.1.

A  $C^1$  submanifold  $\varphi_s$  is said to be an s-slice of size  $(\varepsilon_1, \delta, K)$ , or an s-slice modeled on  $\overline{B_{\varepsilon_1}^x}$ , intersecting  $W_{loc}^u(0)$  transversally at  $(0, y^*)$  with  $|y^*| \leq \delta$  and having the inclination  $\leq K$ , if

$$\varphi_s = \{(x, y) : y = g(x), |x| \leq \varepsilon_1, |y^*| = |g(0)| \leq \delta, g \in C^1 \text{ and } ||Dg|| \leq K\}.$$

A  $C^1$  submanifold  $\varphi_u$  is said to be a u-slice of size  $(\varepsilon_1, \delta, K)$  or a u-slice modeled on  $\overline{B_{\varepsilon_1}^y}$ , intersecting  $W_{loc}^s(0)$  transversally at  $(x^*, 0)$  with  $|x^*| \leq \delta$  and having the inclination  $\leq K$ , if

$$\varphi_u = \{(x, y) : x = h(y), |y| \leq \varepsilon_1, |x^*| = |h(0)| \leq \delta, h \in C^1 \text{ and } ||Dh|| \leq K\}.$$

In all of the above,  $\varepsilon_1, \delta, K$  are positive constants.

Lemma 4.2, 4.4, 4.5, 4.6 are called the Inclination Lemmas and for diffeomorphisms in  $R^n$ , see [2] and [12]. They play the same roles as Lemma 3.3, estimates (3.5) in [14]. However, those estimates are not valid in our case.

Lemma 4.2. Given  $K > 0$ , there exist  $\varepsilon_1, \delta > 0$  and  $c > 1$  such that

- (i) for any u-slice  $\varphi_u$  of size  $(\varepsilon_0/c, \delta, K)$ ,  $\varepsilon_0 \leq \varepsilon_1$ ,  $F$  sends  $\varphi_u$  diffeomorphically onto its image and  $\overline{B_{\varepsilon_0}^y} \cap F(\varphi_u)$  is a u-slice of size  $(\varepsilon_0, \delta, K)$ ;
- (ii) for any s-slice  $\varphi_s$  of size  $(\varepsilon_0/c, \delta, K)$ ,  $\varepsilon_0 \leq \varepsilon_1$ ,  $\overline{B_{\varepsilon_0}^x} \cap F^{-1}(\varphi_s)$  is an s-slice of size  $(\varepsilon_0, \delta, K)$ .

Proof. (i) Let  $F : (x_0, y_0) \rightarrow (x_1, y_1)$ ,  $(x_0, y_0) \in \varphi_u$ . Assume that  $\epsilon$  is small and satisfies

$$(4.4) \quad 1 < d = \frac{1 - \lambda\theta(K+1)}{\lambda}$$

For this  $\theta$ , choose  $\epsilon > 0$  so that  $|Df_1|, |Df_2| \leq \theta$  in  $\overline{B_\epsilon^2}$ . Let  $\delta, \epsilon_1$  satisfy

$$(4.5) \quad \epsilon_1 < \frac{\epsilon}{2}, \quad \delta + K\epsilon_1 < \frac{\epsilon}{2}.$$

Then  $\varphi_u \subset B_\epsilon^2$  and

$$(4.6) \quad \begin{cases} x_1 = A(g(y_0)) + f_1(g(y_0), y_0), \\ y_1 = By_0 + f_2(g(y_0), y_0). \end{cases}$$

$$(4.7) \quad \begin{cases} x_1 = A(g(y_0)) + f_1(g(y_0), y_0), \\ y_1 = By_0 + f_2(g(y_0), y_0). \end{cases}$$

Write (4.7) as

$$(4.8) \quad B^{-1}y_1 = y_0 + B^{-1}f_2(g(y_0), y_0)$$

The Lipschitz constant for  $B^{-1}f_2(g(y_0), y_0)$  as a function of  $y_0$  is bounded by  $\lambda\theta(K+1)$ . By the Implicit Function Theorem, the right hand side of (4.8) defines a diffeomorphism from  $y_0 \in \overline{B_{\epsilon_0/c}^2}$  to  $B^{-1}y_1$ , which covers a ball of radius  $(1 - \lambda\theta(K+1))\epsilon_0/c$ . Therefore,  $y_1$  covers a ball of radius

$$\frac{1 - \lambda\theta(K+1)}{\lambda} \cdot \frac{\epsilon_0}{c} = \frac{d}{c} \cdot \epsilon_0$$

Let  $c$ , asserted in the lemma, be  $c = d$ . Substituting  $y_0$  as a function of  $y_1$  into (4.6), we have a  $u$ -slice  $x_1 = g_1(y_1)$ , modeled on  $\overline{B_{\epsilon_0}^y}$  and transverse to  $W_{loc}^s(0)$  at  $F(g(0), 0) = (g_1(0), 0)$ . Since  $F|_{W_{loc}^s(0)}$  is a contraction,  $g_1(0) \leq g(0) \leq \delta$ . It remains to show that  $\|Dg_1\| \leq K$ .

Let  $(\xi, \eta)$  be a tangent vector to  $\varphi_u$  at  $(x_0, y_0)$ ,  $\eta \neq 0$  and  $\frac{|\xi|}{|\eta|} \leq K$ .  
 Let  $(\xi', \eta') = DF(x_0, y_0)(\xi, \eta)$ ,

$$\begin{aligned} \frac{|\xi'|}{|\eta'|} &= \frac{|(\lambda + f_1 x)\xi + f_1 y \eta|}{|f_2 x \xi + (B + f_2 y)\eta|} \\ (4.9) \quad &\leq \frac{(\lambda + \theta)|\xi| + \|f_1 y\| \cdot |\eta|}{(\lambda - 1 - \theta)|\eta| - \theta|\xi|} \\ &\leq \frac{(\lambda + \theta)K + \|f_1 y\|}{d} \end{aligned}$$

Suppose  $\theta$  is small ( $\epsilon$  is small), then

$$\frac{(\lambda + \theta)K + \theta}{d} \leq K,$$

and (i) is proved if  $\epsilon_1, \delta$  are small so that (4.5) is valid.

(ii) Let

$$(4.10) \quad \epsilon_1 < \frac{\epsilon}{2}, \quad K\epsilon_1 + \delta < \frac{\epsilon}{2}$$

Let  $(x_1, y_1) \in \varphi_s$ , an  $s$ -slice of size  $(\epsilon_1, \delta, K)$ . We look for  $(x_0, y_0)$  such that  $F(x_0, y_0) = (x_1, y_1)$

$$By_0 + f_2(x_0, y_0) = h(Ax_0 + f_1(x_0, y_0))$$

or

$$(4.11) \quad y_0 = -B^{-1}f_2(x_0, y_0) + B^{-1}h(Ax_0 + f_1(x_0, y_0))$$

We use the contraction mapping principle to solve (4.11). Let  $H$  be the set of all the continuous functions from  $\overline{B_{\epsilon_0}^x}$  into  $\overline{B_{\epsilon/2}^y}$  with the distance of any two functions in  $H$  given by the supremum norm. Let  $c > 1$

be such that

$$(4.12) \quad \lambda \varepsilon_0 + \theta \varepsilon_0 \leq \varepsilon_0 / c.$$

The existence of such  $c$  is from (4.3). A continuous function  $F\varphi(\cdot)$  is defined on  $\overline{B_{\varepsilon_0}^x}$  for any  $\varphi \in H$  as

$$(4.13) \quad F\varphi(x) = -B^{-1}f_2(x, \varphi(x)) + B^{-1}h(Ax + f_1(x, \varphi(x))),$$

since  $f_1(0, y) = 0$ ,  $|Ax + f_1(x, \varphi(x))| \leq \lambda \varepsilon_0 + \theta \varepsilon_0 \leq \varepsilon_0 / c$  by (4.12) and  $h$  is defined on  $\overline{B_{\varepsilon_0/c}^x}$ . Furthermore,  $F\varphi \in H$  if

$$(4.14) \quad \lambda \theta \cdot \frac{\varepsilon}{2} + \lambda(K\varepsilon_1 + \delta) \leq \frac{\varepsilon}{2}.$$

The verification of (4.14) uses  $f_2(x, 0) = 0$ , (4.10) and (4.3). We observe that  $F : H \rightarrow H$  is a contraction if  $\theta$  is small. Therefore, there is a unique fixed point of  $F$ , denoted by  $h_0$ . We can show that  $h_0 \in C^1(\overline{B_{\varepsilon_1}^x})$  by using the Implicit Function Theorem locally to solve (4.11) in the neighborhood of  $(x_0, h_0(x_0))$ . We also see that  $h_0(0) \leq h(0) \leq \delta$  since  $F|_W^u$  is an expansion. It remains to check that  $\|Dh_0\| \leq K$ . Suppose  $(\xi, \eta)$  is a nonzero tangent vector to  $F^{-1}\varphi_s$  at  $(x_0, y_0)$ .

$$B\eta + f_{2x} \cdot \xi + f_{2y} \cdot \eta = Dh(\lambda \xi + f_1 x \cdot \xi + f_1 y \cdot \eta),$$

$$|\eta|(1 - \lambda \theta - \lambda K \theta) \leq (\lambda^2 K + \lambda \theta K + \lambda \|f_{2x}\|) |\xi|.$$

$|\xi| = 0$  would imply that  $|\eta| = 0$ , thus  $|\xi| \neq 0$  and

$$(4.15) \quad \begin{aligned} \frac{|\eta|}{|\xi|} &\leq \frac{\lambda^2 K + \lambda K \theta + \lambda \|f_{2x}\|}{1 - \lambda \theta (K+1)} \\ &= \frac{(\lambda + \lambda)K + \|f_{2x}\|}{d} \end{aligned}$$

If  $\theta$  is small ( $\epsilon$  small),  $\frac{(\theta + \lambda)K + \theta}{d} \leq K$  and (ii) is proved. This completes the proof of Lemma 4.2.

Definition 4.3. Let  $\varphi_u^{(1)}, \varphi_u^{(2)}$  be two  $u$ -slices of size  $(\epsilon_1, \delta, K)$  and let  $\varphi_s^{(1)}, \varphi_s^{(2)}$  be two  $s$ -slices of size  $(\epsilon_1, \delta, K)$ . Define the distances with respect to the uniform norm as

$$d(\varphi_u^{(1)}, \varphi_u^{(2)}) = \sup_{|y| \leq \epsilon_1} |g_1(y) - g_2(y)|,$$

$$d(\varphi_s^{(1)}, \varphi_s^{(2)}) = \sup_{|x| \leq \epsilon_1} |h_1(x) - h_2(x)|,$$

where  $\varphi_u^{(i)}, \varphi_s^{(ii)}$  are graphs of  $g_i, h_i, i = 1, 2$ .

Lemma 4.4. Given  $K > 0$ , the constants  $\epsilon_1, \delta$  can be chosen so that the results of Lemma 1 are true. Moreover, there is a constant  $0 < \tilde{\lambda} < 1$  such that

$$d(F^n \varphi_s^{(1)}, F^n \varphi_u^{(2)}) \leq (\tilde{\lambda})^n d(\varphi_u^{(1)}, \varphi_u^{(2)}),$$

$$d(F^{-n} \varphi_s^{(1)}, F^{-n} \varphi_s^{(2)}) \leq (\tilde{\lambda})^n d(\varphi_s^{(1)}, \varphi_s^{(2)}),$$

where  $F^n, F^{-n}$  are abbreviations for  $(\overline{B_{\epsilon_1}^Y} \cap F)^n$  and  $(\overline{B_{\epsilon_1}^X} \cap F^{-1})^n$  which are defined inductively as follows: while applying on a set  $V \subset \mathbb{R}$ ,

$$(\overline{B_{\epsilon_1}^Y} \cap F)^1 V = \overline{B_{\epsilon_1}^Y} \cap F(V), \quad (\overline{B_{\epsilon_1}^X} \cap F^{-1})^1 V = \overline{B_{\epsilon_1}^X} \cap F^{-1}(V),$$

$$(\overline{B_{\epsilon_1}^Y} \cap F)^{n+1} V = \overline{B_{\epsilon_1}^Y} \cap F[(\overline{B_{\epsilon_1}^Y} \cap F)^n V], \quad (\overline{B_{\epsilon_1}^X} \cap F^{-1})^{n+1} V =$$

$$= \overline{B_{\epsilon_1}^X} \cap F^{-1}[(\overline{B_{\epsilon_1}^X} \cap F^{-1})^n V], \quad n \geq 1.$$

Proof. Suppose  $E_1, E_2$  are Banach spaces and  $\varphi \in C^1(E_1, E_2)$ . We define  $ev : C^1(E_1, E_2) \times E_1 \rightarrow E_2$  as  $ev(\varphi, e) = \varphi(e)$ . Let



$$g_t = (t-1)g_2 + (2-t)g_1,$$

$$h_t = (t-1)h_2 + (2-t)h_1, \quad 1 \leq t \leq 2.$$

We first consider

$$x_1 = Ag_t(y_0) + f_1(g_t(y_0), y_0),$$

$$y_1 = By_0 + f_2(g_t(y_0), y_0),$$

or

$$(4.16) \quad x_1 = A \operatorname{ev}(g_t, y_0) + f_1(\operatorname{ev}(g_t, y_0), y_0),$$

$$(4.17) \quad y_1 = By_0 + f_2(\operatorname{ev}(g_t, y_0), y_0).$$

For  $y_1$  fixed,  $y_0$  can be solved as a function of  $t$  in (4.17), and substituted into (4.16) to obtain  $x_1$  as a function of  $t$ . We shall estimate  $\frac{\partial x_1}{\partial t}$  by more symmetric formulas. Assume that  $\delta y_0, \delta x_1, \delta y_1, \delta t$  are tangent vectors in the corresponding spaces, and  $Dg_t$  is the derivative of  $g_t(\cdot)$ . Then,

$$\begin{cases} \delta x_1 = A[\operatorname{ev}(g_2 - g_1)\delta t, y_0] + Dg_t \cdot \delta y_0 \\ \quad + f_{1x} \cdot [\operatorname{ev}((g_2 - g_1)\delta t, y_0) + Dg_t \cdot \delta y_0] + f_{1y} \cdot \delta y_0, \\ 0 = \delta y_1 = B\delta y_0 + f_{2x}[\operatorname{ev}((g_2 - g_1)\delta t, y_0) + Dg_t \cdot \delta y_0] + f_{2y} \cdot \delta y_0. \end{cases}$$

$$\begin{cases} |\delta x_1| \leq (\lambda + \epsilon) [d(\varphi_u^{(1)}, \varphi_u^{(2)}) |\delta t| + K |\delta y_0|] + \lambda \epsilon \cdot |\delta y_0|, \\ |\delta y_0| \leq \lambda \epsilon [d(\varphi_u^{(1)}, \varphi_u^{(2)}) |\delta t| + K |\delta y_0|] + \lambda \epsilon \cdot |\delta y_0|. \end{cases}$$

It follows that

$$\left| \frac{\partial y_0}{\partial t} \right| \leq \frac{\lambda \theta}{1 - \lambda \theta (K+1)} d(\varphi_u^{(1)}, \varphi_u^{(2)}) \leq \frac{\theta}{d} d(\varphi_u^{(1)}, \varphi_u^{(2)}).$$

$$\left| \frac{\partial x_1}{\partial t} \right| \leq (\lambda + \theta) d(\varphi_u^{(1)}, \varphi_u^{(2)}) + ((\lambda + \theta)K + \theta) \cdot \left| \frac{\partial y_0}{\partial t} \right|,$$

Using the estimate for  $|\partial y_0 / \partial t|$ , we find that, when  $\theta$  is small, there exists  $0 < \tilde{\lambda} < 1$  such that

$$\left| \frac{\partial x_1}{\partial t} \right| \leq \tilde{\lambda} d(\varphi_u^{(1)}, \varphi_u^{(2)}).$$

Therefore  $|x_1(y_1, g_2) - x_1(y_1, g_1)| \leq \int_1^2 \left| \frac{\partial x_1}{\partial t} \right| dt \leq \tilde{\lambda} d(\varphi_u^{(1)}, \varphi_u^{(2)})$ . The first inequality in the lemma is proved.

Next consider

$$B y_0 + f_2(x_0, y_0) = h_t(A x_0 + f_1(x_0, y_0))$$

or

$$B y_0 + f_2(x_0, y_0) = \text{ev}(h_t, A x_0 + f_1(x_0, y_0)).$$

Let  $x_0$  be fixed and  $y_0$  be a function of  $t$ ,

$$B \cdot \delta y_0 + f_{2y} \cdot \delta y_0 = \text{ev}((h_2 - h_1) \delta t, A x_0 + f_1(x_0, y_0)) + D h_t \cdot f_{1y} \cdot \delta y_0$$

$$|\delta y_0| (1 - \lambda \theta - \lambda \theta K) \leq \lambda d(\varphi_s^{(1)}, \varphi_s^{(2)}) \cdot |\delta t|.$$

Therefore

$$\left| \frac{\partial y_0}{\partial t} \right| \leq \frac{1}{d} d(\varphi_s^{(1)}, \varphi_s^{(2)}),$$

and  $|y_0(x_0, h_2) - y_0(x_0, h_1)| \leq \int_1^2 \left| \frac{\partial y_0}{\partial t} \right| dt \leq \frac{1}{d} d(\varphi_s^{(1)}, \varphi_s^{(2)})$ . If  $\tilde{\lambda} = \frac{1}{d}$ ,

the second inequality is proved. This completes the proof of the lemma.

Lemma 4.5. Assume further that  $Df_1$  and  $Df_2$  are uniformly continuous in  $\overline{B_\varepsilon^Z}$ . Then, for any  $K > 0$ ,  $\varepsilon_1, \delta$  can be chosen such that for any  $u$ -slice  $\varphi_u$  ( $s$ -slice  $\varphi_s$ ) of size  $(\varepsilon_1, \delta, K)$ ,  $F^n \varphi_u$  is a  $u$ -slice ( $F^{-n} \varphi_s$  is an  $s$ -slice) of size  $(\varepsilon_1, \delta_n, K_n)$ , with  $\delta_n \leq \tilde{\lambda}^n \delta$  and  $K_n \rightarrow 0$  as  $n \rightarrow \infty$ , where  $0 < \tilde{\lambda} < 1$  and  $F^n, F^{-n}$  are abbreviations as before.

Proof. Only  $K_n \rightarrow 0$  has to be proved. Since  $f_{1y}(0, y) = 0$ , by the uniform continuity of  $Df_1$ , for any  $\varepsilon > 0$  there is a  $\xi > 0$  such that  $\|f_{1y}(x, y)\| \leq \varepsilon$  if  $|x| \leq \xi$  and  $|y| \leq \varepsilon_1$ . From Lemma 4.4, there is an  $n_0 > 0$  such that  $F^n \varphi_u \subset \overline{B_\xi^X} \times \overline{B_{\varepsilon_1}^Y}$  for  $n \geq n_0$ . By (4.9), we obtain that  $K_{n+1} \leq \frac{(\lambda + \varepsilon) K_n + \varepsilon}{d}$ ,  $n \geq n_0$ . Thus,

$$K_{n_0+n} \leq \left(\frac{\lambda + \varepsilon}{d}\right)^n K_{n_0} + \frac{\varepsilon}{d - (\lambda + \varepsilon)}.$$

$$\lim_{n \rightarrow \infty} K_n \leq \frac{\varepsilon}{d - (\lambda + \varepsilon)}.$$

Since  $\varepsilon$  is arbitrary, this implies  $K_n \rightarrow 0$  as  $n \rightarrow \infty$ .

A similar proof is applied to  $F^{-n} \varphi_s$ , if we consider  $f_{2x}(x, 0) = 0$ , uniform continuity of  $Df_2$  in a neighborhood of 0, Lemma 4.4 for  $F^{-n} \varphi_s$ , and (4.15). This finishes the proof of the lemma.

The proof of Lemma 4.6 below is similar to that of Lemmas 4.2, 4.4 and 4.5 and shall be omitted. However, due to the lack of uniform continuity of the derivatives, the results concerning  $C^1$  closeness must be formulated very carefully. Let  $M_1$  and  $M_2$  be  $C^1$  submanifolds in  $Z$ . By  $M_2$  is  $C^1$ - $\varepsilon$  near  $M_1$ ,  $\varepsilon$  a positive number, we mean that there are Banach spaces  $E_1, E_2$  such that  $Z = E_1 \oplus E_2$  and a constant  $\rho > 0$  such that  $M_1$  is the graph of  $h_1 : B_\rho^{E_1} \rightarrow E_2$ ,  $i = 1, 2$ , and  $\|h_1 - h_2\|_{C^1(B_\rho^{E_1}, E_2)} \leq \varepsilon$ . Conversely,

questions arise. What is the range of  $J_1$ ? Is  $J_1$  injective? If  $J_1$  is injective, is  $J_1^{-1}$  continuous? The affirmative answer to these questions would ensure that  $TZ$  is homeomorphic to the subspace of sequences of symbols  $(U_i's)$  and  $\Pi F$ , acting on  $TZ$ , is equivalent to the shift operator  $\sigma$  defined on the space of  $\tau_u's$  via  $J_1$ .

Definition 5.1. For  $S = \{U_0, \dots, U_m\}$  and  $\bar{k} > 0$  an integer, a subset  $TU \subset \Pi_N S$  is defined as  $\tau_u \in TU$  if and only if

- 1)  $\tau_u(i) = U_j$  implies that  $\tau_u(i+1) = U_{j+1}$  for  $1 \leq j < m$ ,
- 2)  $\tau_u(i) = U_m$  implies that  $\tau_u(i+j) = U_0$  for  $1 \leq j \leq \bar{k}$ ,
- 3)  $\tau_u(i) = U_1$  implies that  $\tau_u(i+j) = U_0$  for  $1 \leq j \leq \bar{k}$ .

$TU$  is a topological space with the topology induced from  $\Pi_N S$ .

To understand the meaning of this definition, suppose  $\bigcup_{0 \leq i \leq m} U_i$  is a neighborhood of a homoclinic trajectory asymptotic to a fixed point  $0$  of  $F$ . Suppose  $0 \in U_0$ . Then to say  $J_1 \tau_u \in TU$  is equivalent to saying that, if  $\tau(j) \in U_0$  for some  $j$ , then it stays in  $U_0$  for at least  $\bar{k}$  iterates of  $F$  and one can leave  $U_0$  only by going to  $U_1$  and then march back to  $U_0$  staying again for at least  $\bar{k}$  iterates of  $F$ . The same remark applies to  $F^{-1}$ . The main theorem stated below is saying essentially that  $J_1$  is a homeomorphism between  $TZ$  and  $TU$  if  $\bigcup_{0 \leq i \leq m} U_i$  is some neighborhood of a transverse homoclinic orbit.

We are ready to state our main theorem.

Theorem 5.2. Let  $X, Y$  and  $Z = X \times Y$  be Banach spaces,  $F : Z \rightarrow Z$  defined as in Theorem 3.1 with  $DF$  uniformly continuous in a neighborhood of the hyperbolic fixed point of  $F$ . Assume that the local stable and unstable

manifolds are  $W_{loc}^s(0) = \{y = 0\}$ ,  $W_{loc}^u(0) = \{x = 0\}$ , that (4.1), (4.2) are satisfied and  $W_{loc}^u(0) \neq \{0\}$ . Suppose  $\tau_z^\Gamma$  is a homoclinic trajectory and  $\tau_i^\Gamma(z) \rightarrow 0$  as  $i \rightarrow \pm\infty$ . Let  $N > 0$  be an integer with  $\tau_z^\Gamma(-N) \in W_{loc}^u(0)$  and  $\tau_z^\Gamma(N) \in W_{loc}^s(0)$ , where  $W_{loc}^u(0)$  and  $W_{loc}^s(0)$  are contained in  $\overline{B_\varepsilon}$  and (4.3) is valid in  $\overline{B_\varepsilon}$ . Assume that the following conditions are satisfied.

- 1)  $F^{2N}$  sends a disc  $O_1 \subset W_{loc}^u(0)$  centered at  $\tau_z^\Gamma(-N)$  diffeomorphically onto  $O_2 = F^{2N}O_1$ , containing  $\tau_z^\Gamma(N)$ .
- 2)  $O_2 \nparallel W_{loc}^s(0) = \tau_z^\Gamma(N)$ .

Then  $\tau_z^\Gamma$  is a transverse homoclinic trajectory. Furthermore, there exist pairwise disjoint open subsets  $U_0, \dots, U_m$ ,  $m \geq 2$ , in  $Z$ , and an integer  $\bar{k} > 0$  such that  $0 \in U_0$ ,  $O_{\tau_z^\Gamma} \subset \bigcup_{0 \leq i \leq m} U_i$  and such that  $J_1$  is an homeomorphism between  $TZ$  and  $TU$  defined in Definition 5.1.  $\Pi F$  acting on  $TZ$  is equivalent to  $\sigma$  acting on  $TU$  via  $J_1$ .

The open set  $\bigcup_{0 \leq i \leq m} U_i$  is called the extended neighborhood of  $O_{\tau_z^\Gamma}$  with  $U_0$  the "body" and  $\bigcup_{1 \leq i \leq m} U_i$  the "handle".

Before proving Theorem 5.2, we give a symbolization consisting of infinitely many symbols for a subset of  $TU$ .

Definition 5.3.

$$TZ_0 = \{\tau_z : \tau_z \in TZ, \tau_z(0) \in U_0 \text{ and } \tau_z(-1) \in U_m\},$$

$$TU_0 = \{\tau_u : \tau_u \in TU, \tau_u(0) = U_0 \text{ and } \tau_u(-1) = U_m\}.$$

The set  $TZ_0 \cup (TU_0)$  is both open and closed in  $TZ \cup (TU)$ . We observe that

$$\bigcup_i \sigma^i(TZ_0) = TZ \setminus \{(\dots, 0][0, \dots)\}.$$

$$\bigcup_i \sigma^i(TU_0) = TU \setminus \{(\dots, U_0][U_0, \dots)\}.$$

and  $J_1(TZ_0) \subset TU_0$ ,  $TZ_0 \supset J_1^{-1}(TU_0)$ . Therefore,  $J_1 : TZ\{(\dots, 0][0, \dots)\} \rightarrow TU\{(\dots, U_0][U_0, \dots)\}$  is a homeomorphism if and only if  $J_1 : TZ_0 \rightarrow TU_0$  is a homeomorphism, since  $J_1\tau_z = \tau_u$  if and only if  $J_1(\sigma\tau_z) = \sigma\tau_u$  and  $\sigma$  is a homeomorphism on both  $TZ$  and  $TU$ .

Let  $[\bar{k}, +\infty]$  be the space of all the integers  $\geq \bar{k} > 0$  furnished with the discrete topology and compactified by  $+\infty$ . Let  $\Pi_N[\bar{k}, +\infty]$  be the product space. For any  $\tau_{\bar{N}} = (\dots, k_{-i}, \dots, k_{-1}][k_0, \dots, k_j, \dots) \in \Pi_N[\bar{k}, +\infty]$ , a corresponding element  $\tau_u \in TU_0$  is defined as:

- 1)  $\tau_u(\ell) = U_m$  if and only if
  - (A)  $\ell = -\sum_{i=1}^j k_{-i} - jm - 1$ ,  $j = 0, 1, \dots$ , provided  $\ell \neq -\infty$ ;
  - (B)  $\ell = \sum_{i=0}^j k_i + jm + m - 1$ ,  $j = 0, 1, \dots$ , provided  $\ell \neq +\infty$ .
- 2)  $\tau_u(\ell - i) = U_{m-i}$ ,  $0 \leq i \leq m$ , for all  $\ell$  defined by (A) or (B).
- 3)  $\tau_u(i) = U_0$  if not defined by 1) and 2).

Accordingly,  $\tilde{J}_2 : \Pi_N[\bar{k}, +\infty] \rightarrow TU_0$  is defined, continuous and onto.

Definition 5.4. A quotient space  $T\bar{N} = \Pi_N[\bar{k}, +\infty]/\sim$  is defined if  $\tau_{\bar{N}}^{(1)} = (\dots, k_{-i}^{(1)}, \dots, k_{-1}^{(1)}][k_0^{(1)}, \dots, k_j^{(j)}, \dots) \sim \tau_{\bar{N}}^{(2)} = (\dots, k_{-i}^{(2)}, \dots, k_{-1}^{(2)}][k_0^{(2)}, \dots, k_j^{(2)}, \dots)$  means that there exist  $-\infty \leq n_1 \leq -1$  and  $0 \leq n_2 \leq +\infty$  such that  $k_{n_1}^{(j)} = k_{n_2}^{(j)} = +\infty$ ,  $j = 1, 2$ , and  $k_i^{(1)} = k_i^{(2)}$  for  $n_1 < i < n_2$ .

Thus, the map  $J_2 : T\bar{N} \rightarrow TU_0$ ,  $J_2[\tau_{\bar{N}}] = \tilde{J}_2\tau_{\bar{N}}$  is well defined, continuous, injective and onto. It is easy to check that a basis  $\mathcal{B}$  for the topology in  $T\bar{N}$  is

$$\mathcal{B} = \{[B] : B = \{\tau_{\bar{N}} : \tau_{\bar{N}}[-i-1, j+1] \in (\geq k, k_{-i}, \dots, k_{-1}][k_0, \dots, k_j, k)\}\}.$$

where  $k_{-i}, \dots, k_{-1}, k_0, \dots, k_j$  are integers and  $k \geq \bar{k}$  is an integer,  $\geq k$  stands for  $[k, +\infty] \subset [\bar{k}, +\infty]$ .

Theorem 5.5.

$TU_0$  and  $T\bar{N}$  are both compact and Hausdorff.  $J_2$  is a homeomorphism from  $T\bar{N}$  onto  $TU_0$ .

The proof of Theorem 5.5 is elementary and is omitted.

Proof of Theorem 5.2. We first show that when  $O_2$  is small,  $F$  sends  $O_2$  diffeomorphically onto a disc  $\cap W_{loc}^S(0)$  at  $\tau_2^{\Gamma}(N+1)$ . Let  $O_2 = \{(x_0, y_0) : x_0 = g(y_0)\}$  with the inclination  $K_0$ . Consider

$$y_1 = By_0 + f_2(g(y_0), y_0).$$

Since  $f_{2x}(\tau_2^{\Gamma}(N)) = 0$ , for any  $\tilde{\theta} > 0$ , we may let  $O_2$  be sufficiently small so that  $\|f_{2x}(x, y)\| \leq \tilde{\theta}$  in  $O_2$ . Thus,  $\|\frac{df_2}{dy_0}\| \leq K_0\tilde{\theta} + \theta$ , and  $y_0$  can be solved as a  $C^1$  function of  $y_1$  if  $\lambda(K_0\tilde{\theta} + \theta) < 1$ . Substituting into (4.6),  $x_1$  is a  $C^1$  function of  $y_1$ . Therefore, by induction,  $F^i O_2$  contains a  $C^1$  disc  $\cap W_{loc}^S(0)$  at  $\tau_2^{\Gamma}(N+i)$ ,  $i \geq 0$ , with the inclination  $K_i$ , and is diffeomorphic to a disc of  $O_2$ . We give estimates on  $K_i$ 's. Let  $(\xi_i, \eta_i)$ ,  $|\eta_i| \neq 0$  be a tangent vector to a small disc contained in  $F^i O_2$ , on which we assume that  $\|f_{2x}(x, y)\| \leq \tilde{\theta}_i$ .

$$\begin{aligned} \frac{|\xi_{i+1}|}{|\eta_{i+1}|} &= \frac{|(A + f_{1x})\xi_i + f_{1y}\eta_i|}{|f_{2x}\xi_i + (B + f_{2y})\eta_i|} \\ &\leq \frac{(\lambda + \theta)|\xi_i| + \theta|\eta_i|}{(\lambda^{-1} - \theta)|\eta_i| - \tilde{\theta}_i|\xi_i|} \\ &\leq \frac{(\lambda + \theta)K_i + \theta}{d_i}, \end{aligned}$$

where  $d_i = \frac{1 - \lambda(K_i\tilde{\theta}_i + \theta)}{\lambda}$ . There exists a constant  $d_{\infty}$  such that

$d_i \geq d_{\infty} > 1$  for all  $i \geq 0$  provided that the disc contained in  $F^i O_2$  is

sufficiently small, and  $\delta_i$  is sufficiently small, since we have  $\lambda + \epsilon < 1$ .  
Therefore,

$$K_{i+1} \leq \frac{(\lambda + \epsilon)K_i + \dots}{d_\infty},$$

and

$$K_i < K_0 + \frac{\epsilon}{d_\infty - (\lambda + \epsilon)} \stackrel{\Delta}{=} K_\infty, \quad i \geq 0.$$

This completes the proof of the transversality of the homoclinic trajectory  $\tau_z^\Gamma$ .

We next consider  $F^{-2N}W_{loc}^s(0)$  in a neighborhood of  $\tau_z^\Gamma(-N)$ . From Lemma 4.6, it contains a  $C^1$  disc  $\cap W_{loc}^u(0)$  at  $\tau_z^\Gamma(-N)$  and is denoted by  $R_1$ . Analogue to what has been done in Lemma 1.2, we obtain that  $F^{-i}R_1$  contains a disc  $\cap W_{loc}^u(0)$  at  $\tau_z^\Gamma(-N-i)$ ,  $i \geq 0$ , with the inclination  $< K_{1\infty}$  for some constant  $K_{1\infty} > 0$ . The key to the proof is (4.1) and  $|f_{1y}(x,y)|$  being arbitrarily small in some sufficiently small neighborhood of each  $\tau_z^\Gamma(-N-i)$ .

We now construct  $U_0, \dots, U_m$  and  $\bar{k}$  as asserted in the theorem. Suppose that  $\epsilon_1, \epsilon_2$  are positive constants such that for  $u$ -slices of size  $(\epsilon_2, \epsilon_1, k_\infty)$  and  $s$ -slices of size  $(\epsilon_1, \epsilon_2, K_{1\infty})$ , Lemma 4.2-4.6 are valid. Assume that only a finite number of points of  $O_{\tau_z^\Gamma}$ , denoted by  $q_1, \dots, q_{m-1}$ ,  $m \geq 2$ , are outside  $\tilde{U} = B_{\epsilon_1}^x \times B_{\epsilon_2}^y$ . There exist an open neighborhood  $V_i$  for each  $q_i$  such that

$$V_i \cap V_j = \emptyset, \quad 1 \leq i \neq j \leq m-1,$$

$$V_i \cap \tilde{U} = \emptyset, \quad 1 \leq i \leq m-1,$$



$$FV_i \subset V_{i+1}, \quad 1 \leq i \leq m-2$$

$$FV_{m-1} \subset \tilde{U}, \quad F\tilde{U} \cap V_i = \emptyset, \quad 2 \leq i \leq m-1.$$

Let  $p_0, p_1 \in O_{\tau_z} \cap \tilde{U}$ ,  $Fp_1 = q_1$ ,  $Fq_{m-1} = p_0$ , and  $F^m p_1 = p_0$ .

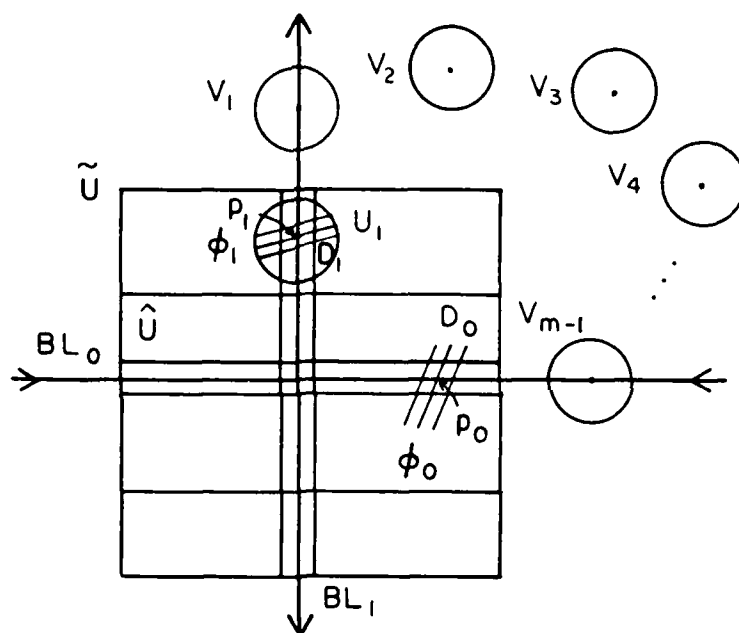


Figure 5.1.

We have shown that it is legitimate to assume that  $F^m W_{loc}^u(0)$  contains a  $C^1$  disc  $\phi_0 \pitchfork W_{loc}^s(0)$  at  $p_0$  with  $\phi_0$  being a  $u$ -slice of size  $(\epsilon_3, \epsilon_1 - \eta, K_\infty - \eta)$  and that  $F^{-n} W_{loc}^s(0)$  contains a  $C^1$  disc  $\phi_1 \pitchfork W^u$  at  $p_1$  with  $\phi_1$  being an  $s$ -slice of size  $(\epsilon_4, \epsilon_2 - \eta, K_{1\infty} - \eta)$  with some constants  $\eta > 0$ ,  $0 < \epsilon_3 < \epsilon_2$ ,  $0 < \epsilon_4 < \epsilon_1$ . By Lemma 4.6, if  $0 < \eta_1 < \eta$ ,  $\epsilon_3, \epsilon_4$  are

small enough, the  $F^m$  image of any  $u$ -slice  $C^1$  near  $W_{loc}^u(0)$  contains a  $u$ -slice  $C^{1-\eta}$ , near  $\varphi_0$ , and hence, a  $u$ -slice of size  $(\varepsilon_3, \varepsilon_1^{-\eta+\eta_1}, K_1)$ , and the  $F^{-m}$  image of any  $s$ -slice  $C^1$  near  $W_{loc}^s(0)$  contains an  $s$ -slice  $C^{1-\eta_1}$  near  $\varphi_1$ , and hence, an  $s$ -slice of size  $(\varepsilon_4, \varepsilon_2^{-\eta+\eta_1}, K_2)$ . We denote the family of all  $u$ -slices of size  $(\varepsilon_3, \varepsilon_1^{-\eta+\eta_1}, K_1)$  by  $U$  and the family of all  $s$ -slices of size  $(\varepsilon_4, \varepsilon_2^{-\eta+\eta_1}, K_2)$  by  $S$ . We may assume that  $\bar{U} \subset \tilde{U}$  and  $\bar{S} \subset \tilde{U}$  in the point set sense. We use  $\bar{W}$  or  $C\bar{W}$  to denote the closure of a set  $\bar{W}$ .

Consider  $BL_1(\bar{k}) = \bigcup_{k \geq \bar{k}} (B_{\varepsilon_2}^Y \cap F)^k \mathcal{U}$  for a positive integer  $\bar{k}$ . When  $\bar{k}$  is large,  $BL_1(\bar{k})$  is  $C^1$  near  $B_{\varepsilon_2}^Y$  and  $F^m(BL_1(\bar{k}))$  is  $C^{1-\eta}$ , near  $\varphi_0$ . Similarly, consider  $BL_0(\bar{k}) = \bigcup_{k \geq \bar{k}} (B_{\varepsilon_1}^X \cap F^{-1})^k S$ . When  $\bar{k}$  is large,  $BL_0(\bar{k})$  is  $C^1$  near  $B_{\varepsilon_1}^X$  and  $F^{-m}(BL_0(\bar{k}))$  is  $C^{1-\eta_1}$  near  $\varphi_1$ . If  $\eta_1$  is small and  $\bar{k}$  is large,  $F^m(BL_1(\bar{k})) \cap BL_0(\bar{k})$ . The intersection is denoted by  $D_0$ . Also  $F^{-m}(BL_0(\bar{k})) \cap BL_1(\bar{k})$  and the intersection is denoted by  $D_1$ . We may assume that  $\bar{D}_0, \bar{D}_1 \subset \tilde{U}$  and  $F\bar{D}_1 \subset V_1$ . It is clear that  $F^m D_1 = D_0$  and  $F^{-m} D_0 = D_1$  if restricted to a neighborhood of  $p_1$ .

It also follows from Lemma 4.6 and 4.2 that if  $\bar{k}$  is large enough,  $F^{m+k}(F^{-m-k})$ ,  $k \geq \bar{k}$  are Lipschitz contractions in the  $C^{(0)}$  norm, on  $u$ -slices in  $U$  into  $u$ -slices near  $\varphi_0$  ( $BL_0(\bar{k})$  into  $BL_0(\bar{k})$ ), with the Lipschitz constant  $\leq \tilde{\lambda}$ ,  $0 < \tilde{\lambda} < 1$ .

Let  $\hat{U} = \tilde{U} \cap F^{-1}\tilde{U}$ . Then  $\hat{U}$  is open and  $p_1 \notin \hat{U}$  since  $q_1 \notin \tilde{U}$ . If  $\bar{k}$  is large, the distance between  $\bar{D}_1$  and  $\hat{U}$  is positive. It is also clear that  $\tilde{U} \cap F^{-1}\hat{U} \subset \tilde{U} \cap F^{-1}\tilde{U} = \hat{U}$ ,  $\tilde{U} \cap F^{-1}\tilde{U} = \hat{U} \cap F^{-1}\tilde{U}$ . By induction, we have  $(\hat{U} \cap F^{-1})^{\bar{k}-1}\hat{U} = (\tilde{U} \cap F^{-1})^{\bar{k}-1}\tilde{U}$ . Clearly,  $D_0 \subset (\tilde{U} \cap F^{-1})^{\bar{k}-1}\tilde{U}$ . We claim that  $\bar{D}_0 \subset (\tilde{U} \cap F^{-1})^{\bar{k}-1}\tilde{U}$ , since  $\bar{U}$  and  $\bar{S} \subset \tilde{U}$ . Therefore,  $\bar{D}_0 \subset (\hat{U} \cap F^{-1})^{\bar{k}-1}\hat{U}$  and  $\bar{D}_1 \subset F^{-m}(\bar{D}_0) \subset F^{-m}(\hat{U} \cap F^{-1})^{\bar{k}-1}\hat{U}$ . The last set is open so there is an

open neighborhood  $U_1$  of  $\bar{U}_1$  such that  $U_1 \cap \hat{U} = \emptyset$ ,  $U_1 \subset \bar{U}$ ,  $FU_1 \subset V_1$  and  $U_1 \subset F^{-m}(\hat{U} \cap F^{-1})^{\bar{k}-1}\hat{U}$ . We claim that  $U_0 = \hat{U}$ ,  $U_1, U_i = V_{i-1}$ ,  $2 \leq i \leq m$ , associated with  $\bar{k}$  ( $U_1$  depends on  $\bar{k}$ ) fulfill all the requirements of the theorem.

We first show that  $J_1(TZ) \subset TU$ . For this, only condition 2) in Definition 5.1 has to be checked. Suppose  $\tau_z \in TZ$  with  $\tau_z(-1) \in U_m$ ,  $\tau_z(0) \in U_0$ , then  $\tau_z(-m) \in U_1 \subset F^{-m}(\hat{U} \cap F^{-1})^{\bar{k}-1}\hat{U}$ . This implies that  $\tau_z(0) \in (\hat{U} \cap F^{-1})^{\bar{k}-1}\hat{U}$ . Hence, for  $1 \leq j \leq \bar{k}-1$ ,  $\tau_z(j) \in (\hat{U} \cap F^{-1})^{k-1-j}\hat{U} \subset \hat{U} = U_0$ . Therefore,

$$J_1(TZ) \setminus \{(\dots, 0][0, \dots)\} \subset TU \setminus \{(\dots, U_0][U_0, \dots)\}.$$

This, together with  $J_1(\dots, 0][0, \dots) = (\dots, U_0][U_0, \dots)$ , implies  $J_1(TZ) \subset TU$ .

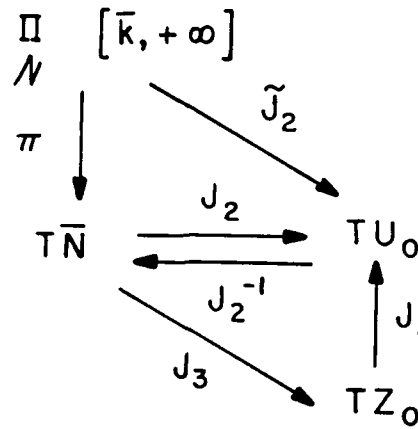
It remains to show that  $J_1(TZ) \supset TU$  and  $J_1^{-1}$  is single valued and continuous. It suffices to prove the following assertions:

- (i)  $J_1^{-1}$  is well defined, single valued and continuous on  $TU \setminus \{(\dots, U_0][U_0, \dots)\}$ ;
- (ii)  $J_1^{-1}(\dots, U_0][U_0, \dots) = (\dots, 0][0, \dots)$  and  $J_1^{-1}$  is continuous at  $(\dots, U_0][U_0, \dots)$ .

For (ii), by Theorem 3.1,  $J_1^{-1}(\dots, U_0][U_0, \dots)$  must lie on  $W_{loc}^s(0)$  and  $W_{loc}^u(0)$ ; hence, identically equal to zero. It follows from Lemma 4.4 that if  $\tau_z \in TZ$  such that  $\tau_z[-i, i] = (\underbrace{U_0, \dots, U_0}_i][\underbrace{U_0, \dots, U_0}_{i+1})$ , then  $\tau_z(0)$  lies on  $s$ -slices  $C(\tilde{\lambda})^i$  near  $W^s$  and  $u$ -slices  $C(\tilde{\lambda})^{i+1}$  near  $W^u$  ( $0 < \tilde{\lambda} < 1$ ) in the  $C^0$  norm. Therefore,  $\tau_z(0)$  is in a ball of radius  $2C(\tilde{\lambda})^i$  centered at 0.  $\tau_z(0) \rightarrow 0$  as  $i \rightarrow \infty$ . Therefore,  $J_1^{-1}$  is continuous at  $(\dots, U_0][U_0, \dots)$ .

For (i), it suffices to show that  $J_3 \triangleq J_1^{-1}J_2$  is well defined, single

valued and continuous on  $T\bar{N}$ , since by Theorem 5.5,  $J_2 : T\bar{N} \rightarrow TU_0$  is a homeomorphism. Also, see the comment after Definition 5.3. It is now clear that we have to show that  $J_1^{-1}\tilde{J}_2$  is well defined, single valued and continuous on  $\prod_N [\bar{k}, +\infty]$ .



Let  $\tau_{\bar{N}} = (\dots, k_{-i}, \dots, k_{-1})[k_0, \dots, k_j, \dots] \in \prod_N [\bar{k}, +\infty]$ . Assume that  $k_n \neq +\infty$  for all  $n$ . The other cases can be proved similarly. If  $\tau_z \in J_1^{-1}\tilde{J}_2 \tau_{\bar{N}}$ , it is necessary that  $\tau_z(0) \in D_0$ . Let  $Z(k_{-i}, \dots, k_{-1})[k_0, \dots, k_j]$ , denote the subset of  $D_0$  such that for each  $z \in Z(k_{-i}, \dots, k_{-1})[k_0, \dots, k_j]$ , there exists a finite trajectory  $\tau_z^0$  with  $J_1 \tau_z^0 = (k_{-i}, \dots, k_{-1})[k_0, \dots, k_j]$  and  $\tau_z^0(0) = z$ . Evidently,

$$F^{k-1+m} Z(k_{-i}, \dots, k_{-2})[k_{-1}, \dots, k_j] = Z(k_{-i}, \dots, k_{-1})[k_0, \dots, k_j].$$

We claim that  $z(k_{-N}, \dots, k_{-1})[k_0, \dots, k_{N-1}]$  is contained in a set of  $s$ -slices  $\subset BL_0(k)$  (a set of  $u$ -slices  $\subset F^m(BL_1(k))$ ) in which the distance between any two of them is  $\leq C(\tilde{\lambda})^N$ . This is clearly true for  $N = 1$ . For  $N = 2$ , the assertion follows from

$$F^{k-1+m} Z(k_{-2})[k_{-1}, k_0, k_1] = Z(k_{-2}, k_{-1})[k_0, k_1],$$

$$F^{k_0+m} Z(k_{-2}, k_{-1})[k_0, k_1] = Z(k_{-2}, k_{-1}, k_0)[k_1],$$

and the contractiveness of  $F^{k-1+m}$  on u-slices and  $F^{-k_0-m}$  on s-slices considered. It follows by induction that the assertion is valid for general  $N$ . We have shown that

$$(5.1) \quad C\ell Z(k_{-N}, \dots, k_{-1})[k_0, \dots, k_{N-1}] \subset \text{a closed ball of radius } \leq C_1(\tilde{\lambda})^N.$$

It is easy to see that  $\tau_z(0) \in \bigcap_{i,j>0} C\ell Z(k_{-i}, \dots, k_{-1})[k_0, \dots, k_j]$ . Similarly,

$$(5.2) \quad \begin{aligned} \tau_z(-\ell) &\in \bigcap_{i,j>0} C\ell Z(k_{-i}, \dots, k_{-n-1})[k_{-n}, \dots, k_j], \\ \ell &= \sum_{\alpha=1}^n k_{-\alpha} + nm, \quad n = 0, 1, \dots \end{aligned}$$

The right hand side of (5.2) is a singleton set since it is the intersection of descending closed sets with estimates (5.1). Therefore  $\tau_z$  is unique if it exists.

Conversely, define  $\tau_z$  formally by (5.2) on a sequence of infinitely many  $-\ell$ 's and choose the values of  $\tau_z$  between each of the  $-\ell$ 's and after  $\tau_z(0)$  by the map  $F$ . We can verify that  $\tau_z$  is a trajectory in  $TZ$  and  $\tilde{J}_2(\dots, k_{-i}, \dots, k_{-1})[k_0, \dots, k_j, \dots) = J_1 \tau_z$ . We start with

$$\begin{aligned} F^{m+k_{-n}} \tau_z(-\ell) &= F^{m+k_{-n}} \bigcap_{i,j>n} C\ell Z(k_{-i}, \dots, k_{-n-1})[k_{-n}, \dots, k_j] \\ &\subset \bigcap_{i,j>n} F^{m+k_{-n}} C\ell Z(k_{-i}, \dots, k_{-n-1})[k_{-n}, \dots, k_j] \\ &\subset \bigcap_{i,j>n} C\ell F^{m+k_{-n}} Z(k_{-i}, \dots, k_{-n-1})[k_{-n}, \dots, k_j] \\ &= \bigcap_{i,j>n} C\ell Z(k_{-i}, \dots, k_{-n})[k_{-n+1}, \dots, k_j] \end{aligned}$$

Since the last is a singleton set, all the inclusions are equalities. This proves the consistency of the definition of  $\tau_z$  on the  $-l$ 's. The only thing unpleasant is that  $\tau_z(-l) \in \text{Cl } Z(k_{-i}, \dots, k_{-n-1}][k_{-n}, \dots, k_j)$ , not  $\tau_z(-l) \in Z(k_{-i}, \dots, k_{-n-1}][k_{-n}, \dots, k_j)$ . But,

$$\begin{aligned} \text{Cl } Z(k_{-i}, \dots, k_{-n-1}][k_{-n}, \dots, k_j) &\subset \text{Cl } Z[k_{-n}, \dots, k_j) \\ &\subset Z[k_{-n}, \dots, k_{j-1}) \end{aligned}$$

due to the continuity of the forward iterates of  $F$ . Therefore, the iterates of  $F$  on  $\tau_z(-l)$  must stay in the "body" for  $k_{-n}, \dots, k_{j-1}$  times before leaving the "body" for the "handle". Since  $j$  can be arbitrarily large,  $\tau_z \in \text{TZ}$  and  $\tilde{J}_2(\dots, k_{-i}, \dots, k_{-1}][k_0, \dots, k_j, \dots) = J_1 \tau_z$ .

The continuity of  $J_1^{-1} \tilde{J}_2$  follows from (5.1). This completes the proof of the theorem.

#### Corollary 5.6.

(Sil'nikov [14])  $\text{T}\bar{\text{N}}$  is homeomorphic to  $\text{TZ}_0$  via the map  $J_3 = J_1^{-1} J_2$ .

#### Corollary 5.7.

Suppose the distance between  $U_i, U_j$ ,  $0 \leq i < j \leq m$  is positive. Let  $J_1 \tau_z^{(\beta)} = \tau_u^{(\beta)}$ ,  $\beta = 1, 2$ . Then,  $\tau_z^{(1)}(i) \rightarrow \tau_z^{(2)}(i)$  as  $i \rightarrow +\infty$  ( $-\infty$ ) if and only if  $\tau_u^{(1)}(i) = \tau_u^{(2)}(i)$  for  $i \geq n$  ( $\leq n$ ), where  $n$  is some constant.

Proof. Necessity is trivial. Sufficiency follows from estimate (5.1).

#### §6. Further Consequences.

Throughout this section, we assume the hypotheses of Corollary 5.7 are satisfied. The above results are generalizations of the work of Sil'nikov [14] on diffeomorphisms in  $\mathbb{R}^n$ . We generalized it to  $C^k$  maps in Banach

spaces, and refined the argument by showing that the extended neighborhood and  $\bar{K}$  can be associated in such a way that all the trajectories in the neighborhood can be symbolized precisely by  $TU$ , depending on  $\bar{K}$ . Note that, in the notation of Sil'nikov's original work, trajectories in  $N^+$ ,  $N^-$ ,  $N^\pm$ , and  $N$ , i.e., asymptotic to  $0$  in the positive direction, negative direction, both directions, and not asymptotic to  $0$  at all, are symbolized distinctly. However, our work shows that trajectories in any of the four subsets are dense, a phenomenon concealed by his original symbolization. To illustrate, we show that the trajectories that are asymptotic to  $0$  in both directions are dense in  $TZ$ . Given  $\tau_z \in TZ$ ,  $J_1 \tau_z = \tau_u = (\dots, u_{x-1}, \dots, u_{x-1}] [u_{x_0}, \dots, u_{x_j}, \dots)$ . Let  $\tau_u^{(n)} = (\dots, u_0, u_0, u_{x-n}, \dots, u_{x-1}] [u_{x_0}, \dots, u_{x_n}, u_0, u_0, \dots)$ ,  $n \geq 1$  and  $\tau_z^{(n)} = J_1^{-1} \tau_u^{(n)}$ . By Corollary 5.7,  $\tau_z^{(n)}$  is asymptotic to  $0$  in both directions for each  $n \geq 1$ . Furthermore,  $\tau_z^{(n)} \rightarrow \tau_z$  since  $\tau_u^{(n)} \rightarrow \tau_u$ .

All the significance of the symbolizations for diffeomorphisms discussed by Sil'nikov and Smale hold true in our case. For example, there are countably many trajectories that are periodic or homoclinic to  $0$  in  $TZ$ .  $TZ$  is topologically transitive, i.e., there is a trajectory  $\tau_z \in TZ$  such that  $\sigma^n \tau_z$ ,  $n = 0, \pm 1, \dots$ , is dense in  $TZ$ . We infer that each trajectory in  $TZ$  is unstable from the instability of  $TU$ , since given any  $\tau_u \in TU$ , we can construct  $\tau_u^{(\ell)}$  such that  $\tau_u^{(\ell)}(-\infty, \ell] = \tau_u(-\infty, \ell]$  and  $\tau_u^{(\ell)}(i) \neq \tau_u(i)$  for infinitely many  $i > \ell$ . From Corollary 5.7,  $\limsup_{i \rightarrow \infty} |\tau_z^{(\ell)}(i) - \tau_z(i)| \geq \varepsilon > 0$ , where  $\tau_z^{(\ell)} = J_1^{-1} \tau_u^{(\ell)}$  and  $\tau_z = J_1^{-1} \tau_u$ ,  $\varepsilon$  is a constant independent of  $\ell$ . But  $\tau_z^{(\ell)}(0) \rightarrow \tau_z(0)$  as  $\ell \rightarrow \infty$ . This proves the instability of each trajectory.

The following is a counterpart to Smale's invariant, Cantor like set near a homoclinic point [16].

Corollary 6.1. There exist an integer  $k > 0$  and a subset of trajectories  $TZ(k)$  of  $F^k$  in a neighborhood of  $0_{\tau_z}$  such that  $F^k$  acting on  $TZ(k)$  is invariant and equivalent to the shift of the dynamic system of two symbols.

Proof. By Theorem 5.2, it suffices to examine  $\sigma^k$  on  $TU$ . Let  $k \geq \bar{k} + m$  be any fixed integer. If, by the symbol  $s_0$ , we mean  $\{U_0, \dots, U_0\}$  and the  $k$ -fold symbol  $s_1, \{U_0, \dots, U_0, U_1, \dots, U_m\}$ , a subset of  $TU$  is defined and is invariant under  $\sigma^k$ .

Comparing our results with other papers, one finds that the invariant set of trajectories under  $\Pi F$  are discussed instead of the invariant set of points under  $F$ . For  $F$  being diffeomorphic, define  $P$  as the projection  $P: TZ \rightarrow Z$ ,  $P\tau_z = \tau_z(0)$ . Then  $P$  is a homeomorphism from  $TZ$  onto  $TZ(0) \stackrel{\text{def}}{=} P(TZ)$ .  $\Pi F: TZ \rightarrow TZ$  is equivalent to  $F: TZ(0) \rightarrow TZ(0)$ , via  $P$ .

$$\begin{array}{ccc} TZ & \xrightarrow{\Pi F} & TZ \\ \downarrow P & & \downarrow P \\ TZ(0) & \xrightarrow{F} & TZ(0) \end{array}$$

Therefore, the symbolizations for the point set  $TZ(0)$ , invariant under  $F$  is induced from that of  $TZ$ , or  $F: TZ(0) \rightarrow TZ(0)$  is equivalent to a shift homeomorphism  $\sigma: TU \rightarrow TU$ .

Another interesting case is the appearance of a snap-back repeller named after Marotto [10]. An expanding fixed point  $0$  of a  $C^1$  map  $F: Z \rightarrow Z$  is said to be a snap-back repeller if there is a point  $z_0 \in W_{loc}^u(0)$  with  $z_0 \neq 0$ , and an integer  $n \geq 1$  such that  $F^n(z_0) = 0$  and  $DF^i(z_0)$  is an isomorphism onto  $Z$ , for  $1 \leq i \leq n$ . It is easy to see that there is a transverse homoclinic trajectory  $\tau_z^\Gamma$  passing through  $z_0$  and hitting  $0$



after finite iterates of  $F$ . And it can be treated as a special case of Theorem 5.2 with  $W_{loc}^s(0) = \{0\}$ . However, the results are nicer if we consider positive trajectories  $\tau_z^+$  and  $\tau_u^+$ . Let  $U_0, \dots, U_m$  be open sets containing  $0_{\tau_z^+}$  and  $0 \in U_0$ . Let  $S = \{U_0, \dots, U_m\}$  and  $\bar{k} > 0$  an integer. A subset  $TU^+ \subset \Pi_N^+ S$  is defined on  $\tau_u^+ \in TU^+$  if and only if 1) and 2) but 3) of Definition 5.1 hold.  $TU^+$  is a topological space with the topology induced from  $\Pi_N^+ S$ . The semishift operator  $\sigma^+$  is defined on  $TU^+$  as  $\sigma^+ \tau_u^+(i) = \tau_u^+(i+1)$ ,  $i \in \mathbb{N}^+$ .  $\sigma^+$  is continuous, surjective but not injective. Let  $TZ^+ \subset \Pi_N^+ Z$  be the set of all the positive trajectories whose orbits are contained in  $\bigcup_{0 \leq i \leq m} U_i$ .  $TZ^+$  is a topological space with the topology induced from  $\Pi_N^+ Z$ . Let  $P$  be the projection from  $TZ^+$  to  $TZ^+(0) \stackrel{\text{def}}{=} P(TZ^+) \subset Z$ , defined as  $P\tau_z^+ = \tau_z^+(0) \in Z$  for any  $\tau_z^+ \in TZ^+$ . It is obvious that  $P$  is a homeomorphism. Let  $J_1 : TZ^+ \rightarrow TU^+$  be defined as  $\tau_u^+(i) = (J_1 \tau_z^+)(i) = U_j$  if  $\tau_z^+(i) \in U_j$ ,  $0 \leq j \leq m$ ,  $i \geq 0$ .

Theorem 6.2. Suppose  $F : Z \rightarrow Z$  is  $C^1$  with  $0$  as a snap-back repeller.

Then there exist open sets  $U_0, \dots, U_m$  and an integer  $\bar{k} > 0$  such that

$\bigcup_{0 \leq i \leq m} U_i$  contains the homoclinic orbit and  $0 \in U_0$ . Furthermore,  $J_1 : TZ^+ \rightarrow TU^+$  is a homeomorphism and the following diagram commutes.

$$\begin{array}{ccc}
 TZ^+(0) & \xrightarrow{F} & TZ^+(0) \\
 P \uparrow & & \uparrow P \\
 TZ^+ & \xrightarrow{\Pi F} & TZ^+ \\
 J_1 \downarrow & & \downarrow J_1 \\
 TU^+ & \xrightarrow{\sigma^+} & TU^+
 \end{array}$$

The proof of Theorem 6.2 is similar to that of Theorem 5.2. One only has to observe that the  $s$ -slices are points in  $Z$  and the  $u$ -slices coincide with  $W_{loc}^u(0)$ . We don't ask that  $DF$  be uniformly continuous in the neighborhood

of 0 since the Inclination Lemmas are trivially true in this case. We obtain that, when a snap-back repeller appears, the above symbolic dynamics can be used to discuss trajectories, positive trajectories and invariant point sets in a neighborhood of the homoclinic orbit.

### §7. Chaotic behavior.

We have shown that trajectories in  $TZ$  have very complicated behavior - the motion of  $F^i \tau_z(0)$  is quite unpredictable except that it must stay in  $U_0$  for at least  $\bar{k}$  iterates of  $F$  before leaving  $U_0$  for the "handle". We shall show that this kind of motion implies chaos described by Li and Yorke [9], [10], [17]; that is, if  $TZ$  is homeomorphic to  $TU$  via  $J_1$ , then there exists chaos in the following sense:

1) There exists  $k > 0$  such that for each integer  $p \geq k$ ,  $F$  has a trajectory of period  $p$ .

2) There exists a subset of uncountably many trajectories  $CHAOS \subset TZ$  such that,

a) for every  $\tau_z^{(1)}, \tau_z^{(2)} \in CHAOS$  with  $\tau_z^{(1)} \neq \tau_z^{(2)}$ ,

$$(7.1) \quad \limsup_{i \rightarrow +\infty} |\tau_z^{(1)}(i) - \tau_z^{(2)}(i)| > 0;$$

b) for  $\tau_z^{(1)} \in CHAOS$  and  $\tau_z^{(2)}$  being periodic in  $TZ$ , (7.1) is valid.

c)  $\tau_z^{(1)}, \tau_z^{(2)} \in CHAOS$  implies that

$$\liminf_{i \rightarrow +\infty} |\tau_z^{(1)}(i) - \tau_z^{(2)}(i)| = 0.$$

3)  $\Pi F(CHAOS) = CHAOS$ .

The ideas of the proof presented here are essentially from [9], [10].

Proof. 1) Let  $k = \bar{k} + m$ . Let  $\tau_u = (\dots, \text{repeat}, \underbrace{U_0, \dots, U_0, U_1, \dots, U_1}_{p\text{-fold}}, \text{repeat}, \dots)$

then  $\tau_z = J_1^{-1} \tau_u \in T\mathbb{Z}$  is a trajectory of  $F$  with the period =  $p$ .

2) Let  $s_0 = \underbrace{\{U_0, \dots, U_0\}}_{k\text{-fold}}$ ,  $s_1 = \underbrace{\{U_0, \dots, U_0, U_1, \dots, U_1\}}_{k\text{-fold}}$ ,  $k = \bar{k} + m$ .

For each  $w \in (0,1)$ , choose an element  $\tau_u^w \in TU$ , composed by  $s_0$  and  $s_1$  such that

$$\tau_u^w \in \{(\dots; s_{\alpha_{-i}}; \dots; s_{\alpha_{-1}})[s_{\alpha_1}; s_{\alpha_2}; \dots; s_{\alpha_j}; \dots) : \\ \alpha_i = 1 \text{ only if } i = \pm n^2, n=1,2,\dots; \text{ and } \lim_{T \rightarrow \infty} \frac{R^+(\tau_u^w, n^2)}{n} = w\},$$

where  $R^+(\tau_u^w, n^2)$  is the number of  $\alpha_i$ 's which equals 1 for  $(1 \leq i \leq n^2) (-n^2 \leq i \leq -1)$  respectively.

Let  $CHS = \{\sigma^i \tau_u^w : w \in (0,1), i \in \mathbb{N}\}$ . Evidently,  $\sigma(CHS) = CHS$ . Therefore, if  $CHAOS = J_1^{-1}(CHS)$ ,  $\Pi F(CHS) = CHS$ . The assertion 3) is proved. In proving 2), we only consider the case  $i \rightarrow +\infty$ . We first show that a) is true for  $\tau_z^{(1)} = J_1^{-1}(\tau_u^w)$  and  $\tau_z^{(2)} = J_1^{-1}(\sigma^j \tau_u^w)$ ,  $j \neq 0$ . Since  $w \neq 0$ , there exist infinitely many integers  $n$  such that  $\tau_u^w(kn^2-1) = U_m$ .  $\sigma^j \tau_u^w(kn^2-1) = \tau_u^w(kn^2+j-1)$ . If  $n$  is sufficiently large,  $kn^2+j-1$  is not of the form  $kl^2-1$  for any integer  $l$ , thus  $\sigma^j \tau_u^w(kn^2) \neq U_m$ . This shows (7.1) is valid in this case. Obviously a) is also true for  $\tau_z^{(1)} = J_1^{-1}(\sigma^i \tau_u^w)$ ,  $\tau_z^{(2)} = J_1^{-1}(\sigma^j \tau_u^w)$ ,  $i \neq j$ . We next show that a) is true for  $\tau_z^{(1)} = J_1^{-1}(\sigma^i \tau_u^{w_1})$  and  $\tau_z^{(2)} = J_1^{-1}(\sigma^j \tau_u^{w_2})$ ,  $w_1 \neq w_2$ . Let  $\tilde{R}^+(\tau_u, kn^2)$  be the number of  $\tau_u(\cdot)$  which equals  $U_m$  for  $1 \leq i \leq kn^2$ . We observe that

$$(7.2) \quad \lim_{n \rightarrow \infty} \frac{\tilde{R}^+(\sigma^i \tau_u^w, kn^2)}{n} = w.$$

For any given  $K > 0$ , there exists an  $\lambda > K$  such that  $\tau_u^{(1)} \neq \tau_u^{(2)}$ . Otherwise, from (7.2), one would have  $w_1 = w_2$ , contradicting the fact that  $w_1 \neq w_2$ . The proof of a) is completed. b) can be proved similarly. To prove c), notice that for any  $\tau_u \in \text{CHS}$ , the length of the successive  $i$ 's such that  $\tau_u(i) = U_0$  approaches  $+\infty$  as  $i \rightarrow +\infty$ . Therefore, for  $\tau_u^{(1)}$ ,  $\tau_u^{(2)} \in \text{CHS}$ , the length of successive  $i$ 's such that  $\tau_u^{(1)}(i) = \tau_u^{(2)}(i) = U_0$  approaches  $+\infty$  as  $i \rightarrow +\infty$ . c) is true by (5.1). This completes the proof of the existence of chaos.

The work of Li and Yorke indicated that Period 3 implies chaos in  $\mathbb{R}$ . Marotto pointed out this is not the case in  $\mathbb{R}^2$ . He proved that Snap-back Repeller implies chaos in  $\mathbb{R}^n$ . Our work shows that the transverse homoclinic trajectory implies chaos in Banach spaces.

#### §8. Flows.

Noninvertible maps also arise from the Poincaré mapping of noninvertible flows. The Poincaré map can either be the return map around a periodic trajectory for an autonomous flow or the period map of a periodic flow. Both cases are discussed in this section.

Let  $X$  be a Banach space and  $T(t,s)$ ,  $t \geq s$  in  $\mathbb{R}$  be a semigroup of nonlinear maps in  $X$ . We assume that

- 1)  $T(t,s)$  is strongly continuous in  $t,s$ ;
- 2)  $T(s,s) = I$ ;
- 3)  $T(t,u)T(u,s) = T(t,s)$ ,  $t \geq u \geq s$ ;
- 4) There are constants  $\alpha \geq 0$ ,  $k \geq 1$  such that  $T(t,s)x$  is  $C^k$  jointly in  $t$  and  $x$  for  $t \geq s + \alpha$ .

Examples of abstract evolution equations with  $\alpha = 0$  may be found in [4].

For delay equations under some general conditions,  $\alpha = k\gamma$ , where  $\gamma > 0$  is the delay [3].

We say that  $T(t,s)$  is periodic of period  $\omega > 0$  if  $T(t,s) = T(t+\omega, s+\omega)$ . If we do not assume that  $\omega$  is the least period, then we may assume  $\omega > \alpha$ . The period map  $F = T(\omega, 0)$  is then  $C^k$  on  $X$ . If  $\xi(t)$  is a periodic trajectory of  $T(t,s)$  with the period  $\omega$ ; that is,  $T(t,s)\xi(s) = \xi(t)$ ,  $t \geq s$  in  $\mathbb{R}$ ,  $\xi(t+\omega) = \xi(t)$ , then  $\xi(0)$  is a fixed point of  $F$ . Conversely, any fixed point of  $F$  can be used to define a periodic trajectory. One can define homoclinic trajectories of  $T(t,s)$  asymptotic to  $\xi(t)$  in the obvious way and relate them to homoclinic trajectories of  $F$  asymptotic to  $\xi(0)$ .

We next assume that the semigroup is autonomous; i.e.,  $T(t,s) = T(t-s)$ ,  $t \geq s$  in  $\mathbb{R}$ . Let  $\xi(t)$  be a periodic trajectory of least period  $\omega > 0$  of  $T(t)$ ,  $t \geq 0$ ; that is,  $T(t)\xi(s) = \xi(t+s)$  for all  $t \geq 0$ ,  $s \in \mathbb{R}$ ,  $\xi(t+\omega) = \xi(t)$  for all  $t$  and  $\xi(t) \neq \xi(0)$ ,  $0 < t < \omega$ . Replacing  $\omega$  by  $n\omega$ , we may assume  $\omega > \alpha$ . Let  $X_1 \subset X$  be a codimension one hyperplane transversal to the periodic trajectory at  $x = \xi(0)$ . There exists a neighborhood  $U$  of  $\xi(0)$  in  $X_1$  such that for every  $x \in U$ , there is a unique  $t = t(x)$  near  $\omega$  such that  $T(t(x))x \in X_1$ . The map  $F : U \rightarrow X_1$  is defined as  $F(x) = T(t(x))x$  and is  $C^k$ . It is clear that  $\xi(0)$  is a fixed point of  $F$ . Suppose  $x = p(t)$  is a homoclinic trajectory of  $T(t)$  asymptotic to  $x = \xi(t)$ . There is a constant  $\tau > \frac{\alpha}{2}$  such that for  $|t| > \tau$ ,  $x = p(t)$  is near the orbit of  $x = \xi(t)$  and intersects  $U \subset X_1$  successively as  $t \rightarrow \pm\infty$ . Let  $q_1 = p(t_1)$  and  $q_2 = p(t_2)$ ,  $q_1, q_2 \in U$ , with  $t_1 < -\tau$  and  $t_2 > \tau$ .  $F^{-n}q_1$  and  $F^nq_2$ ,  $n \geq 0$ , are defined as the intersections of  $p(t)$  with  $X_1$  and agree with the definition of  $F$  given before. Obviously,  $F^nq_2 \rightarrow \xi(0)$  and  $F^{-n}q_1 \rightarrow \xi(0)$  as  $n \rightarrow \infty$ . Assume that there are open sets  $U_1$  and  $U_2 \subset U$

such that  $U_1 \cap U_2 = \emptyset$ ,  $q_1 \in U_1$  and  $(\bigcup_{n=0}^{\infty} F^n q_2) \cup (\bigcup_{n=1}^{\infty} F^{-n} q_1) \subset U_2$ . We redefine  $F$  in  $U_1$  as  $Fq_1 = q_2$  and  $Fx = y$  for  $x \in U_1$  and  $y \in U_2$  such that  $u = T(t(x))x$  with a unique  $t = t(x)$  near  $t_2 - t_1 > \epsilon$ . This could be done if  $U_1$  is sufficiently small so that the flow issuing from  $U_1$  meets  $U_2$  transversely in a uniquely determined time  $\hat{t} = t(x)$  near  $t_2 - t_1$ . Thus,  $F : U_1 \cup U_2 \rightarrow X_1$  is  $C^k$  with a fixed point  $\xi(0)$  and a homoclinic trajectory  $\{F^{-n}q_1, F^nq_2, n \geq 0\}$ .

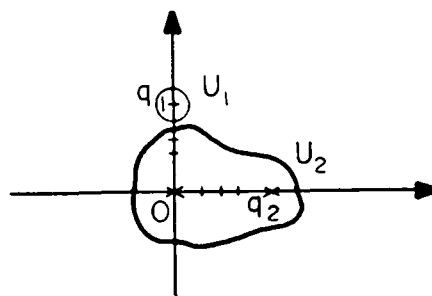


Figure 8.1

Definition 8.1. Suppose  $T(t,s)$  satisfies hypertheses 1) - 4) and is either periodic or autonomous. Suppose that  $x = (t)$  is a periodic trajectory with the Poincaré map  $F$  defined previously. It is said to be a hyperbolic periodic trajectory if  $\sigma(DF(\xi(0))) \cap \{|\lambda| = 1\} = \emptyset$ .

Note that the map  $F$  can be different if we take other hyperplanes transversal to the periodic trajectory, e.g., in the periodic flow case, the section can be  $\{t^*\} \times X \subset \mathbb{R} \times X$  and the map is  $T(t^* + \omega, t^*)$ . Thus, we shall justify that Definition 8.1 is independent of the Poincaré section chosen. Also, if  $\omega < \alpha$ , there is no unique way to choose  $n\omega > \epsilon$  with integers  $n > 0$ . We shall prove Definition 8.1 is independent of  $n$ .

The stable set  $W^S(\xi(\cdot))$  and unstable set  $W^U(\xi(\cdot))$  of  $x = \xi(t)$  is defined in the usual way. The existence of the local stable manifold  $W_{loc}^S(\xi(\cdot)) \subset W^S(\xi(\cdot))$  and local unstable manifold  $W_{loc}^U(\xi(\cdot)) \subset W^U(\xi(\cdot))$  in a neighborhood of the orbit of a hyperbolic periodic trajectory  $x = \xi(t)$  shall be proved in Theorem 8.3.

Definition 8.2. A homoclinic trajectory  $x = p(t)$  of  $T(t,s)$  in a Banach space  $X$ , asymptotic to a periodic trajectory  $x = \xi(t)$  of  $T(t,s)$  is said to be a transverse homoclinic trajectory if

- 1) the periodic trajectory  $x = \xi(t)$  is hyperbolic;
- 2) for any sufficiently large pair  $s, t > 0$  such that  $p(-s) \in W_{loc}^U(\xi(\cdot))$  and  $p(t) \in W_{loc}^S(\xi(\cdot))$ ,  $T(t,-s)$  sends a disc containing  $p(-s)$  in  $W_{loc}^U(\xi(\cdot))$  diffeomorphically onto its image which is transversal to  $W_{loc}^S(\xi(\cdot))$  at  $p(t)$ .

Note that in the forgoing definitions  $W_{loc}^S(\xi(\cdot)) = \{\xi(\cdot)\}$  as well as  $x = p(t)$  hits the orbit  $O_{\xi(\cdot)}$  at some finite  $t$  is allowed. It is also clear that  $W_{loc}^U(\xi(0))$  and  $W_{loc}^S(\xi(0))$  of the fixed point  $\xi(0)$  of  $F$  are precisely the intersections of  $W_{loc}^U(\xi(\cdot))$  and  $W_{loc}^S(\xi(\cdot))$  with the Poincaré section. Another observation is that  $x = p(t)$  is a transverse homoclinic trajectory if and only if it induces a transverse homoclinic trajectory on the Poincaré section for the fixed point  $\xi(0)$  of the map  $F$ . There is a geometric explanation for Definition 8.2, that is, there are two narrow strips locally diffeomorphic to  $W_{loc}^U(\xi(\cdot))$  and  $W_{loc}^S(\xi(\cdot))$  respectively (Immersed image of  $W_{loc}^U(\xi(0)) \times \mathbb{R}$  and  $W_{loc}^S(\xi(0)) \times \mathbb{R}$ , not necessarily injective), attached to  $x = p(t)$  and intersect transversely along  $x = p(t)$ . See Figure 8.2 for the illustration of the unstable strip.

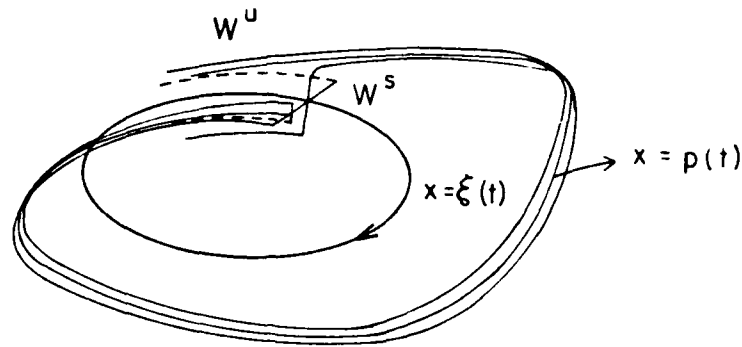


Figure 8.2.

Theorem 8.3. Let  $x = \xi(t)$  be a periodic trajectory with the period  $\omega > 0$ ,  
for  $T(t,s)$  satisfying conditions 1) - 4). Then in both the following cases,  
 $T(t,s) = T(t-s)$  or  $T(t,s) = T(t+\omega, s+\omega)$ , the definition of the hyperbolicity  
of  $x = \xi(t)$  is independent of the integer  $n$ ,  $n\omega > \alpha$ , or the Poincaré section  
chosen. Moreover if  $T(t,s)x$  is  $C^k$  jointly in  $t, s$  and  $x$  for  $t > s + \alpha$ ,  
the local stable and unstable manifolds  $W_{loc}^s(\xi(\cdot))$  and  $W_{loc}^u(\xi(\cdot))$  exist  
and are  $C^k$  submanifolds in  $X$  for the autonomous case and in  $R \times X$  for  
the periodic case.

Proof. Only the proof for the periodic flow shall be given. Let  $F_1 = T(n_1\omega, 0)$ ,  
 $F_2 = T(n_2\omega, 0)$  where  $n_1$  and  $n_2$  are integers with  $n_1\omega > \alpha$ ,  $n_2\omega > \alpha$ .  
 $F_1^{n_2} = F_2^{n_1} \stackrel{\text{def}}{=} F_3$ .  $\xi(0)$  is a hyperbolic fixed point of  $F_3$  if and only if it  
 is a hyperbolic fixed point of  $F_1$  and  $F_2$ . This shows that the definition  
 of the hyperbolicity is independent of the way the period is multiplied.



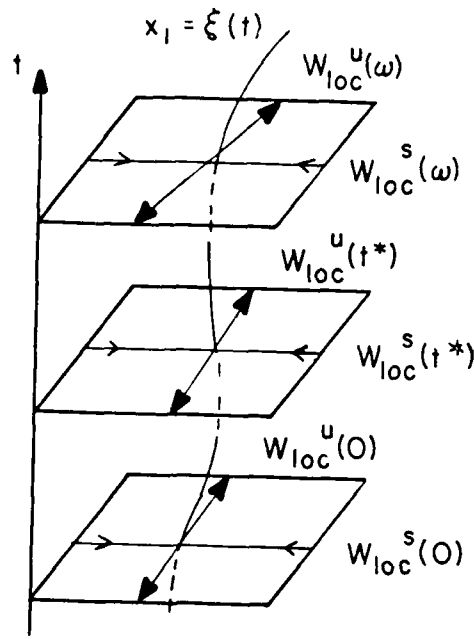


Figure 8.3.

Assume that  $T(\xi, 0)$  has  $\xi(0) = \xi(\omega)$  as a hyperbolic fixed point. The existence of the local  $C^k$  stable and unstable manifolds  $W_{loc}^s(0)$  and  $W_{loc}^u(0)$  of  $T(\omega, 0)$  on the section  $\{0\} \times X \subset \mathbb{R} \times X$  follow from Theorem 3.1. Periodicity implies that  $W_{loc}^u(\omega) = W_{loc}^u(0)$  and  $W_{loc}^s(\omega) = W_{loc}^s(0)$ . Take a section  $\{t^*\} \times X$  and, without loss of generality, assume that  $\alpha < t^*$  and  $\alpha < \omega - t^*$ . Let the stable and unstable sets for  $x = \xi(t)$ ,  $W^s(\xi(\cdot))$  and  $W^u(\xi(\cdot))$ , intersect  $\{t^*\} \times X$  in  $W^s(t^*)$  and  $W^u(t^*)$ . Obviously,  $W^u(t^*) = T(t^*, 0)W^u(0)$  and  $W^s(t^*) = [T(\omega, t^*)]^{-1}W^s(0)$ . It is easy to show that  $T(t^*, 0)$

is a  $C^k$  embedding from  $W_{loc}^u(0)$  into  $W^u(t^*)$  with  $[T(\omega, 0)]^{-1}T(\omega, t^*)$  as the inverse. Therefore  $W_{loc}^u(t^*) \stackrel{\text{def}}{=} T(t^*, 0)W_{loc}^u(0)$  is a  $C^k$  submanifold in  $\{t^*\} \times X$  and  $W_{loc}^u(\omega) = T(\omega, t^*)W_{loc}^u(t^*)$ . Also  $TW_{loc}^u(\omega) = DT(\omega, t^*)TW_{loc}^u(t^*)$ . Now let  $Y \subset X$  be such that  $DT(\omega, t^*) \cdot Y \subset TW_{loc}^S(\omega)$ .  $Y$  is a linear closed subset since  $TW_{loc}^S(\omega)$  is. It is easy to see that  $Y \oplus TW_{loc}^u(t^*) = X$ . We write  $x \in X$  as  $x = (x_1, y_1)$  where  $x_1 \in TW_{loc}^u(t^*)$  and  $y_1 \in Y$ , and use the Implicit Function Theorem to solve  $T(\omega, t^*)W_{loc}^S(t^*) \subset W_{loc}^S(\omega)$ . We obtain that

$$W_{loc}^S(t^*) = \{\xi(t^*) + (x_1, y_1) : x_1 = g(y_1), g \in C^k(B_y^\varepsilon), g(0) = 0, Dg(0) = 0\}$$

for some  $\varepsilon > 0$ . Thus,  $W_{loc}^S(t^*)$  is a  $C^k$  submanifold in  $\{t^*\} \times X$  and  $TW_{loc}^S(t^*) = Y$ . The proof of the invariance of  $W_{loc}^S(t^*)$  and  $W_{loc}^u(t^*)$  under  $T(t^* + \omega, t^*)$  is easy and is omitted. Estimates for the spectra of  $DT(t^* + \omega, t^*)$  on  $TW_{loc}^S(t^*)$  and  $[DT(t^* + \omega, t^*)]^{-1}$  on  $TW_{loc}^u(t^*)$  can be obtained by considering  $[T(t^* + \omega, t^*)]^n = T(t^*, 0) \cdot [T(\omega, 0)]^{n-1} \cdot T(\omega, t^*)$  and  $[T(t^* + \omega, t^*)|_{W_{loc}^u(t^*)}]^{-n} = [T(\omega, t^*)|_{W_{loc}^u(t^*)}]^{-1} [T(\omega, 0)|_{W_{loc}^u(0)}]^{-n+1} \cdot [T(t^*, 0)|_{W_{loc}^u(0)}]^{-1}$  and using  $|\sigma(L)| \leq \lim_{n \rightarrow \infty} (||L^n||)^{\frac{1}{n}}$  for a linear bounded operator  $L$ . Consequently,  $\xi(t^*)$  is a hyperbolic fixed point under  $T(t^* + \omega, t^*)$  and  $W_{loc}^u(t^*)$ ,  $W_{loc}^S(t^*)$  are precisely the local unstable and stable manifolds under  $T(t^* + \omega, t^*)$ , due to the uniqueness. Thus, the definition of the hyperbolicity for the periodic trajectory of flows is independent of the cross sections chosen.

The local unstable set of  $x = \xi(t)$  is a neighborhood of  $t = t^*$  is determined by

$$W_{loc}^u(\xi(\cdot)) = \{(t, T(t, 0)x) : t \in (t^* - \varepsilon, t^* + \varepsilon), x \in W_{loc}^u(0)\} \subset \mathbb{R} \times X,$$

for some  $\varepsilon > 0$ . It is clearly a  $C^k$  submanifold modeled on  $R \times TW_{loc}^u(0)$ . The local stable set of  $x = \xi(t)$  in a neighborhood of  $t = t^*$  is determined by

$$W_{loc}^s(\xi(\cdot)) = \{(t, y) : T(\omega, t)y \in W_{loc}^s(\omega), t \in (t^* - \varepsilon, t^* + \varepsilon)\} \subset R \times X,$$

for some  $\varepsilon > 0$ . Using the local coordinates  $R \times TW_{loc}^u(t^*) \times Y$ , and the Implicit Function Theorem, one shows that  $W_{loc}^s(\xi(\cdot))$  is a  $C^k$  submanifold modeled on  $R \times Y = R \times TW_{loc}^s(t^*)$ . The proof of Theorem 8.3 is completed.

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